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A Statistical Generalization of the Quantum Mechanics (I).

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Summary. — It is shown that the ordinary Schrödinger equation for a dynamical system Σ may be replaced by a more general equation, which has the form of the Schrödinger equation of a quantized Bose field whose quanta are the systems Σ. The linear wave functionals of the quantized field describe pure states of the system Σ and the non linear wave functionals describe, in general, mixtures of states of Σ. The representation in which the emission operators of the quantized Σ field are diagonal plays a central role in the present formalism. It is shown that the eigenfunctionals of the absorption operators of the quantized Σ field can be used to obtain a new description of the states of a system Σ, the expectation values of the field quantities in suitably chosen eigenstates of the emission operators coinciding with the expectation values of the corresponding quantities in the pure states of Σ, but the fluctuations being larger in the former case. The eigenfunctionals of the absorption operators have the remarkable property of being matrix elements of the unity operator of the field formalism, and seem to be of a more fundamental nature than the linear functionals which correspond to the ordinary description of the pure states by means of the wave functions of Σ.

1. – Introduction.

In the ordinary form of the quantum mechanics, the states of a system are described by wave functions $\Psi(t; x)$ depending on the time t and variables x. The time evolution of the wave function is determined by the Schrödinger equation

$$ih\frac{\partial}{\partial t}\Psi(t; x) = H\Psi(t; x)\,, \tag{1}$$

H being the hamiltonian operator of the system Σ. For the sake of simplicity we shall treat only the case of continuous variables x, so that the x will be coordinates of a point in a continuous space Ω. The case, in which some or all of the x are discrete variables, can be treated in a similar way, with minor modifications.

In general, we shall assume that the wave functions $\Psi(x)$ are normalizable and normalized as usual: $\int_{\Omega} |\Psi|^2 \, \mathrm{d}x = 1$. These normalizable wave functions may be considered as vectors of a Hilbert space, the ψ-space. The complex conjugate functions $\Psi^*(x)$ are vectors of another Hilbert space, the ψ^*-space. We have shown [1] that it is possible to describe the motion of the system Σ by means of the vectors of the Hilbert space dual to the ψ^*-space, the χ-space. The vectors of the χ-space correspond to the continuous linear functionals $\chi[\psi^*]$ defined in the ψ^*-space:

$$\chi[\psi^*] = \int_{\Omega} \Psi(x)\psi^*(x) \, \mathrm{d}x \, . \tag{2}$$

It follows from (1) and (2) that the linear functionals $\chi[\psi^*]$ built with the solutions of (1) satisfy the equation.

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \chi[t; \psi^*] = \mathscr{H}\chi[t; \psi^*] = \int_{\Omega} \mathrm{d}x \, \psi^*(x) H \frac{\delta}{\delta\psi^*(x)} \chi[t; \psi^*] \, , \tag{3}$$

which was already given in reference (1). Equation (3) may be considered as the Schrödinger equation of the dual χ-space. Equation (3) admits as solutions non linear functionals, which are no more equivalent to wave functions Ψ. It is preferable to consider (3) as the basic equation, instead of (1), because the functionals $\chi[\psi^*]$ allow us to describe, in an effective non symbolic way, states of Σ which can only be described by means of symbolic functions, such as the Dirac δ-function. It is well known from the Schwarz theory of distributions that, in order to avoid the use of symbolic functions, it is necessary to use functionals.

Once it is assumed that (3) is the fundamental equation of evolution, it becomes natural to consider also the non linear functionals which are solutions. In this paper we shall consider only the solutions of (3) which can be expanded in Volterra series:

$$\chi[\psi^*] = \chi[0] + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \int_{\Omega} \Psi_n(x_1, \ldots, x_n)\psi^*(x_1) \ldots \psi^*(x_n) \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \, . \tag{4}$$

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[1] M. SCHÖNBERG: *Nuovo Cimento*, **10**, 350 (1953).

The $\Psi_n(t; x)$ are solutions of the equations.

$$(5) \qquad i\hbar \frac{\partial}{\partial t} \Psi_n(t; x_1,..., x_n) = \sum_{l=1}^{n} H(x_l)\Psi(t; x_1,..., x_n) .$$

These equations have the same form as the ordinary Schrödinger equations for assemblies of non interacting systems Σ. The Ψ_n are symmetrical with respect to the n sets of variables x.

The functionals (4) may be considered as vectors of a linear space which contains as a sub-space the dual of the ψ^*-space. This extended χ-space can be made into a Hilbert space by introducing the following definition of the inner product of two functionals χ:

$$(6) \qquad (\chi, \chi') = \{\chi[0]\}^*\chi'[0] + \sum_{n=1}^{\infty} \int_{\Omega} \{\Psi_n(x_1,..., x_n)\}^*\Psi'_n(x_1,..., x_n)\, dx_1 ... dx_n .$$

In the special case of linear functionals

$$(6a) \qquad (\chi, \chi') = \int_{\Omega} \Psi^*(x)\Psi'(x)\, dx ,$$

so that the inner product of two linear functionals coincides with the inner product of the corresponding wave functions Ψ. The definition of the inner product leads to a definition of the lengh $\sqrt{(\chi, \chi)}$ of a vector χ and to the normalization condition:

$$(7) \qquad (\chi, \chi) = |\chi[0]|^2 + \sum_{n=1}^{\infty} \int_{\Omega} |\Psi_n|^2\, dx_1 ... dx_n = 1 .$$

Equation (3) coincides with the Schrödinger equation of the second quantization of the systems Σ, treated as bosons, in the representation in which the emission operators $\psi^*_{op}(x)$ are diagonal. From the commutation rules

$$(8) \qquad [\psi_{op}(x), \psi^*_{op}(x')] = \delta(x - x') , \qquad [\psi_{op}(x), \psi_{op}(x')] = [\psi^*_{op}(x), \psi^*_{op}(x')] = 0 ,$$

it follows that it is possible to diagonalize simultaneously all the $\psi^*_{op}(x)$. They are satisfied by taking:

$$(9) \qquad \psi_{op}(x) = \frac{\delta}{\delta \psi^*(x)} .$$

The Schrödinger equation of the second quantization

$$(10) \qquad i\hbar \frac{d\chi}{dt} = \int_{\Omega} \psi^*_{op}(x) H \psi_{op}(x)\, dx\, \chi ,$$

has the same form as (3), in the representation in which the $\psi^*_{op}(x)$ are diagonal. The physical content of the second quantization equation of evolution and of equation (3) are nevertheless different, because the second quantization is a treatment of the many body problem whereas equation (3) corresponds to a single system Σ.

We have shown [2] that the second quantization can be used as a formalism for the statistical quantum mechanics of a system Σ, treated as a quantum of the quantized field obtained by the second quantization of its Schrödinger equation. The Volterra series (4) becomes the Fock expansion in the second quantization and, by means of the functions Ψ_n, it is possible to build a density operator $R(t)$ for a single system Σ with the wave functional $\chi[t; \psi^*]$ of the second quantization:

$$\langle x' | R(t) | x'' \rangle = \sum_{n=1}^{\infty} \int_{\Omega} \Psi_n(t; x', x_1, \ldots, x_{n-1}) \Psi_n^*(t; x'', x_1, \ldots, x_{n-1}) \, \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} . \tag{11}$$

It can be seen that $R(t)$ satisfies the von Neumann equation of motion of the density operators:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} R(t) = [H, R(t)] . \tag{12}$$

In the case of a linear functional χ of the type (2), we have

$$\langle x' | R(t) | x'' \rangle = \Psi(t; x') \Psi^*(t; x'') \tag{13}$$

and $R(t)$ describes the same pure state as the wave function Ψ. Since our formalism for the motion of the system Σ is the same as that of the second quantization, we can apply the same method of definition of density operators $R(t)$ to the solution of (3). *Thus we get a possible interpretation of the non linear solutions of* (3) *as describing, in general, mixtures of states of* Σ.

A non linear solution of (3) may sometimes correspond to a $R(t)$ which describes a pure state of Σ. Thus the denstiy matrix corresponding to the non linear functional $\Delta[\psi^*, \Psi]$

$$\Delta[\psi^*, \Psi] = \exp\left[\int_{\Omega} \psi^*(x) \Psi(x) \, \mathrm{d}x \right] , \tag{14}$$

is:

$$\langle x' | R_\Delta | x'' \rangle = \Psi(x') \Psi^*(x'') \left(\exp\left[\int_{\Omega} |\Psi|^2 \, \mathrm{d}x \right] - 1 \right) \Big/ \int |\Omega|^2 \, \mathrm{d}x , \tag{15}$$

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[2] M. SCHÖNBERG: *Nuovo Cimento*, **10**, 697 (1953).

It was shown in reference (1) that a normalized linear functional χ defines a special kind of probability distribution of the rays of the ψ-space, in which a probability $|\chi[\psi^*]|^2$, with a normalized ψ, is assigned to the ray determined by the vector ψ. The ordinary interpretation rule of the quantum mechanics for discrete non degenerate eigenvalues was obtained by assuming that the probability of finding a value A' of the physical quantity (A) in a measurement in the state of Σ described by the normalized linear functional χ is given by the probability $|\chi[\psi^*_{A'}]|^2$ of the ray determined by the eigenfunction $\psi_{A'}$. We shall see in section 7 that the above result can be easily extended to degenerate discrete eigenvalues or to continuous eigenvalues, with suitable modifications. Thus the probability of obtaining a degenerate discrete value A' is given by the probability of the linear manifold of the eigenvectors of A corresponding to the eigenvalue A', the probability of a linear manifold being taken as the sum of the probabilities of the rays corresponding to a complete set of orthogonal vectors of the manifold.

In the case of non linear functionals, there are difficulties in the introduction of a probability distribution by means of $|\chi[\psi^*]|^2$. The normalization of the vector ψ does not determine completely ψ, since there remains always an undetermined constant phase factor. The value of $|\chi[\psi^*]|^2$ depends in general on the choice of the constant phase factor. Nevertheless in the case of the homogeneous functionals $\chi_n[\psi^*]$

$$\chi_n[\psi^*] = \frac{1}{\sqrt{n!}} \int_{\Omega} \Psi_n(x_1, \ldots, x_n)\psi^*(x_1) \ldots \psi^*(x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n \,, \tag{16}$$

$|\chi_n[\Psi^*]|^2$ does not depend on the choice of the constant phase factor in the normalized ψ. In the second quantization interpretation of the χ-formalism, we are led to take $|\chi[\Psi^*]|^2$ as the probability of finding the values $\Psi(x)$ of the $\psi_{op}(x)$, by applying to the non hermitian operators ψ_{op} the ordinary rule for the computation of expectation values. In this case there is no reason to take only normalized functions $\Psi(x)$ and the value of $|\chi[\Psi^*]|^2$ may be larger than 1. This point will be discussed in section 8.

As long as we consider only linear functionals χ, the replacement of the wave function $\Psi(x)$ by the corresponding linear functional (2) may be considered as a change of representation:

$$\chi[\psi^*] = \int_{\Omega} \langle \psi^* | x \rangle \Psi(x)\, \mathrm{d}x \,, \qquad \langle \psi^* | x \rangle = \psi^*(x) \,. \tag{17}$$

Notwithstanding the fact that now we have a transformation functional $\langle \psi^* | x \rangle$, instead of the ordinary transformation functions, the change of representation defined by (17) has properties which correspond to those of the ordinary

changes of representation. In order to use the new representation, it is necessary to introduce an integration with respect to the functional variable ψ^*. We shall take:

$$\int \{\chi[\psi^*]\}^* \chi'[\psi^*] \mathrm{d}\psi^* = (\chi[\psi^*], \chi'[\psi^*]), \tag{18}$$

$\{\chi[\psi^*]\}^*$ must be considered as a functional of the function ψ, instead of ψ^*, in the same way as the complex conjugate of an analytic function $f(z)$ of the complex variable z is an analytic function of z^*. We shall write:

$$\chi^*[\psi] = \{[\psi^*]\}^* . \tag{19}$$

In our integration with respect to ψ^*, the integrated quantity is to be considered as a functional of both ψ and ψ^*, the integration being analogous to a contraction of an upper index with an equal lower index in the tensor calculus. Since

$$\Psi(x) = (\langle \psi^* | x \rangle, \chi[\psi^*]) , \tag{20}$$

we have:

$$\Psi(x) = \int \langle x | \psi \rangle \chi[\psi^*] \mathrm{d}\psi^* , \tag{21}$$

$$\langle x | \psi \rangle = \{\langle \psi^* | x \rangle\}^* = \psi(x) . \tag{22}$$

We may say that the quantity (ψ) has the value ψ when the system Σ is in the state described by the wave function ψ, ψ being taken normalized. With this convention, the ordinary rule of interpretation of the transformation functions is still applicable: $|\langle \psi^* | x \rangle|^2 \mathrm{d}x$ *is the probability of obtaining values of the* (x) *in the ranges* $x - x + \mathrm{d}x$, *in the state in which the quantity* (ψ) *has the value* ψ.

In the representation in which the wave function of Σ becomes the linear functional $\chi[\psi^*]$, the physical quantity (A) of Σ is described by the operator $\mathscr{A}$

$$\mathscr{A} = \int_\Omega \mathrm{d}x \, \psi^*(x) A \frac{\delta}{\delta \psi^*(x)} = \int_\Omega \mathrm{d}x \, \psi^*_{\mathrm{op}}(x) A \psi_{\mathrm{op}}(x) , \tag{23}$$

since we must have

$$\mathscr{A} \chi[\psi^*] = \int_\Omega \langle \psi^* | x \rangle A \Psi(x) \, \mathrm{d}x = \int_\Omega \psi^*(x) A \Psi(x) \, \mathrm{d}x , \tag{24}$$

and it results from (23) that:

$$\mathscr{A} \int_\Omega \psi^*(x) \Psi(x) \, \mathrm{d}x = \int_\Omega \psi^*(x) A \Psi(x) \, \mathrm{d}x . \tag{25}$$

$\mathcal{A}$ is the well known operator for the quantity (A) in the second quantization. Let us introduce the projection operator on the direction of the normalized vector φ in the ψ-space:

$$p_\varphi \psi(x) = \varphi(x) \int_\Omega \varphi^*(x')\psi(x')\,\mathrm{d}x' . \tag{26}$$

The χ-operator corresponding to p_φ will be denoted by N_φ:

$$N_\varphi = \int_\Omega \psi^*_{\mathrm{op}}(x) p_\varphi \psi_{\mathrm{op}}(x)\,\mathrm{d}x = \int_\Omega \psi^*_{\mathrm{op}}(x)\varphi(x)\,\mathrm{d}x \int_\Omega \varphi^*(x')\psi_{\mathrm{op}}(x')\,\mathrm{d}x' . \tag{27}$$

We have

$$N_\varphi = a^*_\varphi a_\varphi , \qquad a_\varphi = \int_\Omega \varphi^*(x)\psi_{\mathrm{op}}(x)\,\mathrm{d}x , \tag{27a}$$

and since

$$[a_\varphi, a^*_\varphi] = \int_\Omega \varphi^*(x)\varphi(x')\left[\psi_{\mathrm{op}}(x), \psi^*_{\mathrm{op}}(x')\right]\mathrm{d}x\,\mathrm{d}x' = \int_\Omega |\varphi|^2\,\mathrm{d}x = 1 , \tag{28}$$

the eigenvalues of N_φ are the non negative integers: 0, 1, 2,... . In the second quantization, N_φ is the operator for the number of systems Σ in the state φ. The definition of a_φ can be applied even to a non normalized φ and we have quite generally:

$$[a_\varphi, a_{\varphi'}] = [a^*_\varphi, a^*_{\varphi'}] = 0 , \qquad [a_\varphi, a^*_{\varphi'}] = \int_\Omega \varphi^*(x)\varphi'(x)\,\mathrm{d}x . \tag{29}$$

The simplest of the operators A is the unity operator of the ψ-space, the corresponding χ-operator will be denoted by N_{op}:

$$N_{\mathrm{op}} = \int_\Omega \psi^*_{\mathrm{op}}(x)\psi_{\mathrm{op}}(x)\,\mathrm{d}x . \tag{30}$$

N_{op} is the operator for the total number of systems Σ in the second quantization. Let the φ_λ be a complete set of orthonormal functions and the p_λ the corresponding projection operators: $p_\lambda = p_{\varphi_\lambda}$. We shall denote the N_{φ_λ} by N_λ. Since

$$\sum_\lambda p_\lambda = 1 , \tag{31}$$

we have:

$$N_{\mathrm{op}} = \sum_\lambda N_\lambda . \tag{32}$$

It follows from the commutation rules (29) that the N_λ are commutable, therefore the eigenvalues of N_{op} are the non negative integers. The linear functionals χ are eigenfunctionals of N_{op} corresponding to the eigenvalue 1. More generally, the homogeneous functionals χ_n defined by (16) are eigenfunctionals of N_{op} corresponding to the eigenvalue n:

$$N_{op}\chi_n[\psi^*] = \int_\Omega \mathrm{d}x\, \psi^*(x) \frac{\delta \chi_n[\psi^*]}{\delta \psi^*(x)} = n\chi_n[\psi^*]\,. \tag{33}$$

The operators $\mathscr{A}$ were derived from the A by the standard methods of the transformation theory, for linear functionals. In the case of non linear functionals, it is often convenient to describe the physical quantity (A) by an operator $\mathscr{A}_{nor}$ which we shall now define. Let us consider the projection operators of the eigenvalues of N_{op}:

$$N_{op}P_n = nP_n\,, \qquad P_nP_{n'} = \delta_{n,n'}P_n\,, \qquad \sum_{n=0}^{\infty} P_n = 1\,. \tag{34}$$

The spectral decomposition of N_{op} is:

$$N_{op} = \sum_{n=1}^{\infty} nP_n\,. \tag{35}$$

Let us introduce the operator L:

$$L = \sum_{n=1}^{\infty} n^{-1}P_n\,, \qquad LN_{op} = N_{op}L = 1 - P_0\,. \tag{36}$$

The operator $\mathscr{A}_{nor}$ is:

$$\mathscr{A}_{nor} = L\mathscr{A} = \mathscr{A}L\,, \qquad (\mathscr{A}P_n = P_n\mathscr{A})\,, \tag{37}$$

$$\mathscr{A}\chi[\psi^*] = \int_\Omega \psi^*(x) A \frac{\delta \chi[\psi^*]}{\delta \psi^*(x)}\,\mathrm{d}x = \tag{38}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \int_\Omega \psi^*(x_1)\dots\psi^*(x_n)\{A_{x_1} + \dots + A_{x_n}\}\Psi_n(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n\,,$$

$$\mathscr{A}_{nor}\,\chi[\psi^*] = \tag{39}$$

$$= \sum_{n=1}^{\infty} \frac{n^{-1}}{\sqrt{n!}} \int_\Omega \psi^*(x_1)\dots\psi^*(x_n)\{A_{x_1} + \dots + A_{x_n}\}\,\Psi_n(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n\,.$$

It results from (38) and (39) that for any linear functional χ_1 we have $\mathscr{A}\chi_1 = \mathscr{A}_{nor}\,\chi_1$.

We have seen that in the χ-formalism the ψ behave as values of a physical quantity, even when only the linear functionals are considered. When the non linear functionals are taken into account, there are eigenfunctionals of the $\psi_{op}(x)$ corresponding to the eigenvalues $\Psi(x)$. Indeed, we have

$$\psi_{op}(x)\,\Delta[\psi^*, \Psi] = \frac{\delta}{\delta\psi^*(x)} \exp\left[\left|\int_\Omega \psi^*(x)\Psi(x)\,dx\right|\right] = \Psi(x)\,\Delta[\psi^*, \Psi]\,, \tag{40}$$

Δ being the functional defined by (14). We have seen that, with the R density matrix interpretation of the χ-formalism, the functional $\Delta[\psi^*, \Psi]$ describes the pure state of Σ of wave function Ψ. We shall see in section 8 that, in the second quantization, $\Delta[\psi^*, \Psi]$ with a normalized Ψ describes a state of the quantized ψ-field in which the expectation value of N_{op} is 1, the expectation value of N_φ being $\left|\int_\Omega \varphi^*(x)\Psi(x)\,dx\right|^2$ and that of any operator $\mathcal{A}$ being $\int_\Omega \Psi^*(x)A\Psi(x)\,dx$, i.e. the expectation value of A in the state Ψ of Σ. It follows from (14) and (18) that for any functional $\chi[\psi^*]$ we have:

$$\chi[\psi^*] = \int \Delta[\psi^*, \Psi]\,\chi[\Psi^*]\,d\Psi^*\,. \tag{41}$$

Equation (41) shows that any state χ of the quantized ψ-field can be obtained by the superposition of states $\Delta[\psi^*, \Psi]$ in which the $\psi_{op}(x)$ have the values Ψ.

The properties of the states of the quantized ψ-field described by the wave functionals $\Delta[\psi^*, \Psi]$ are discussed in section 8. The $\Delta[\psi^*, \Psi]$ with normalized Ψ do not describe one quantum state, because $\Delta[\psi^*, \Psi]$ is not an eigenfunctional of N_{op}, but the expectation values of both N_{op} and N_Ψ are 1. In the field state $\Delta[\psi^*, \Psi]$, the probabilities of the eigenvalues of N_φ are given by a Poisson law with average value equal to the expectation value $\left|\int_\Omega \Psi^*\varphi\,dx\right|^2$ of N_φ. This expectation value of N_φ for a normalized Ψ coincides with the probability of the value 1 of p_φ in the Σ-state Ψ. More generally, the expectation value of $\mathcal{A}$ in the field state $\Delta[\psi^*, \Psi]$ coincides with the expectation value $\int_\Omega \Psi^*A\Psi\,dx$ of the quantity (A) in the Σ-state Ψ, for a normalized Ψ. The probabilities of the eigenvalues of N_{op} are given by a Poisson law with an average value 1. *The above results show that the state of the quantized ψ-field in which the $\psi_{op}(x)$ have the values $\Psi(x)$, i.e. the field state $\Delta[\psi^*, \Psi]$, when Ψ is normalized has an average behaviour coinciding with that of the Σ-state Ψ or of the one quantum field state* $\int_\Omega \Psi(x)\psi^*(x)dx$.

The χ-formalism for a single system Σ can be related to the theory of the quantized ψ-field. Thus the linear wave functional (2) coincides with the wave functional of the one quantum state of the ψ-field in which the quantum Σ is in the state Ψ. The relation is more complicated in the case of non linear functionals. It is discussed in section **9**. The R density operator interpretation of the χ-formalism amounts to consider only the field states without a vacuum part, i.e. field states described by $\chi[\psi^*]$ such that $\chi[0] = 0$, and to represent the physical quantities (A) by the field operators $\mathcal{A}_{\text{nor}}$. It is interesting to notice that the spectrum of $\mathcal{A}_{\text{nor}}$ is dense, with the possible exception of the eigenvalue 0 which may be isolated, and covers densely the interval between the minimum eigenvalue A'_{min} and the maximum eigenvalue A'_{max} of A. The interpretation of the χ-formalism by means of the density operators R amounts to apply the ψ-field formalism to the computation of the expectation values of the operators $\mathcal{A}_{\text{nor}}$. This interpretation is consistent, but does not seem to be the most fundamental, because different wave functionals describe the same mixed or pure Σ-state. It is shown in section **5** that any pure or mixed state of Σ can be described by a quadratic functional $\chi_2[\psi^*]$.

It seems likely that the χ-formalism for a single system Σ be a more complete theory than the ordinary form of the quantum mechanics. The different wave functionals which correspond to the same density operator R may describe different conditions of the system Σ which appear as identical in the degree of approximation in which the description given by the ordinary quantum mechanics is correct. We may think that the ordinary description corresponds to some kind of time average over very small intervals of time, and perhaps over small regions of space also. With a higher degree of accuracy than is now experimentally possible, the description given by the present quantum mechanics may appear as incomplete, different physical conditions being lumped into a single quantum state.

Although the χ-formalism be applicable to any system, it seems likely that its true significance will appear in the cases in which the systems Σ are the elementary particles of the present quantum mechanics. In a more refined description, the present elementary particles will probably appear as complex structures which have an average behaviour of a particle, but whose detailed behaviour in very short intervals of time may be rather different from that of a particle. The functionals $\Delta[\psi^*, \Psi]$ are obviously a more fundamental element of the χ-formalism than the linear functionals and, with a normalized Ψ, they describe quasi-one quantum states, which are quasi-particle states in the cases in which the system Σ is a particle of the ordinary quantum mechanics. The expectation values of the operators $\mathcal{A}$ in a state $\Delta[\psi^*, \Psi]$ coincide with the expectation values of the ordinary quantum mechanics, when Ψ is normalized, but there is a considerable mean square deviation in a state $\Delta[\psi^*, \Psi]$. It seems reasonable to assume that the expectation values cor-

respond to some kind of time average of the instantaneous values and that the fluctuations correspond to deviations from the particle picture. The large fluctuations corresponding to the Poisson distributions for the values of N_{op} and the N_{φ} are particularly remarkable. The spread of the values of the position variables would give rise to a kind of spatial extension of the point particles.

The dense spectrum of the $\mathscr{H}_{nor}$ is also a remarkable property. It may indicate that the quantization of the values of the physical quantities arises from the averaging which introduces the particles.

The representation of the second quantization in which the ψ^*_{op} are diagonal was already discussed by Fock [3]. In Fock's treatment $\psi^*(x)$ is expanded in a series of orthonormal functions φ^*_λ

$$\psi^*(x) = \sum_\lambda a^*_\lambda \varphi^*_\lambda , \tag{42}$$

and the wave functional $\chi[\psi^*]$ is considered as a function of a countable infinity of complex variables a^*. Dirac [4] has developped the Fock treatment by introducing a definition of the inner product involving contour integration with respect to the complex variables a^*. It is more convenient to define the inner product of two functions $f_1(z_1,..., z_r)$ and $f_2(z_1,..., z_r)$ as follows:

$$(f_1(z_1,..., z_r), f_2(z_1,..., z_r)) = \tag{43}$$

$$= \pi^{-r} \int_{-\infty}^{+\infty} \{f_1(z_1,..., z_r)\}^* f_2(z_1,..., z_r) \exp\left[-\sum_{l=1}^{r} |z_l|^2 \right] d\alpha_1 ... d\alpha_r d\beta_1 ... d\beta_r ,$$

$$\alpha_l = \mathcal{R}ez_l , \qquad \beta_l = \mathcal{I}mz_l . \tag{44}$$

The above definition of the inner product can be easily extended to functions of a countable infinity of complex variables; it leads to the definition (6) of the inner product of two functionals and to a more general definition of the $d\psi^*$ integration than (18). Indeed we get from (4):

$$\chi[\psi^*] = A_0 + \sum_{n=1}^{\infty} \sum_\lambda A_{\lambda_1,...,\lambda_n} a^*_{\lambda_1} ... a^*_{\lambda_n} , \tag{45}$$

$$A_0 = \chi[0] , \qquad A_{\lambda_1,...,\lambda_n} = \frac{1}{\sqrt{n!}} \int_\Omega \Psi_n(x_1,..., x_n) \varphi^*_{\lambda_1}(x_1) ... \varphi^*_{\lambda_n}(x_n) dx_1 ... dx_n . \tag{46}$$

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[3] V. Fock: *Zeits. f. Phys.*, **49**, 339 (1928); *Phys. Zeits. Sow. Un.*, **6**, 428 (1934).
[4] P. A. M. Dirac: *Comm. Dublin Inst. f. Adv. Stud.*, A, **1** (1943).

By applying the definition (43) of the inner product extended to a countable infinity of complex variables we get:

$$(47) \qquad (\chi, \chi') = A_0^* A_0' + \sum_{n=1}^{\infty} n! \sum_{\lambda_1,\ldots,\lambda_n} A^*_{\lambda_1,\ldots,\lambda_n} A'_{\lambda_1,\ldots,\lambda_n} =$$

$$= \chi^*[0]\chi'[0] + \sum_{n=1}^{\infty} \int_{\Omega} \Psi_n^*(x_1,\ldots, x_n)\Psi_n'(x_1,\ldots, x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n .$$

In the ordinary treatment of the second quantization, the N_λ are diagonalized and the wave functional becomes a function of the eigenvalues N'_λ of the N_λ. In this representation the inner product of two functionals is taken as follows:

$$(48) \qquad (\chi(N'), \chi'(N')) = \sum_{N'} \chi^*(N')\chi'(N') .$$

We shall see in section 3 that this definition is also equivalent to (6).

The replacement of the wave function Ψ by a functional $\chi[\psi^*]$ corresponds to a generalization of the field model and can also be applied to other field theories, to the classical electromagnetic theory for instance. The χ-formalism is a technique for the treatment of non coherent fields, as well as of random fields, and it is not restricted to the quantum mechanics. The standard method of dealing with non coherent fields corresponds to the introduction of some kind of density matrix; the χ-formalism has the advantage of giving a direct generalization of the field quantities, instead of the bilinear combinations of field quantities of the density matrix method. A general discussion of the generalization of the field theories by means of functionals will be given elsewhere.

2. – The χ-formalism.

Equation (41) shows that the functional Δ is the analogue of the Dirac δ-function in the theory of the functionals χ:

$$(49) \qquad \Delta[\psi^*, \psi'] = \langle \psi^* | 1 | \psi' \rangle, \qquad \Delta[\psi^{*\prime}, \psi] = \{ \Delta[\psi^*, \psi'] \}^* .$$

It follows from (41) that:

$$(50) \qquad \int \Delta[\psi^*, \Psi]\Delta[\Psi^*, \psi']\mathrm{d}\Psi^* = \Delta[\psi^*, \psi'] .$$

Hence

$$(51) \qquad \int \Delta[\psi^*, \Psi]\Delta[\Psi^*, \psi]\mathrm{d}\Psi^* = \Delta[\psi^*, \psi] = \exp\left[\int_{\Omega} |\psi|^2 \mathrm{d}x \right],$$

so that $\Delta[\psi^* \Psi]$ is normalizable, although non normalized.

It results from (41) that

$$\int \psi^*(x)\, \Delta[\psi^*, \Psi]\, \chi[\Psi^*]\, \mathrm{d}\Psi^* = \psi^*(x)\chi[\psi^*] = \psi^*_{\mathrm{op}}(x)\, \chi[\psi^*]\,, \tag{52}$$

and by taking into account (40) we get:

$$\int \Delta[\psi^*, \Psi]\, \Psi(x)\chi[\Psi^*]\, \mathrm{d}\Psi^* = \int \left\{ \frac{\delta}{\delta\psi^*(x)}\, \Delta[\psi^*, \Psi] \right\} \chi[\Psi^*]\, \mathrm{d}\Psi^* = \tag{53}$$

$$= -\frac{\delta}{\delta\psi^*(x)} \int \Delta[\psi^*, \Psi]\, \chi[\Psi^*]\, \mathrm{d}\Psi^* = \frac{\delta}{\delta\psi^*(x)}\, \chi[\psi^*] = \psi_{\mathrm{op}}(x)\, \chi[\psi^*]\,.$$

It follows from (52) and (53) that:

$$\langle \psi'^* | \psi^*_{\mathrm{op}}(x) | \psi''(x) \rangle = \psi'^*(x)\, \Delta[\psi'^*(x), \psi''(x)]\,, \tag{54}$$

$$\langle \psi'^* | \psi_{\mathrm{op}}(x) | \psi''(x) \rangle = \psi''(x)\, \Delta[\psi'^*(x), \psi''(x)]\,. \tag{55}$$

Since

$$\int \left\{ \Delta[\psi^*, \Psi] \int_\Omega \psi^*(x) A \Psi(x)\, \mathrm{d}x \right\} \chi[\Psi^*]\, \mathrm{d}\Psi^* = \tag{56}$$

$$= \int_\Omega \psi^*_{\mathrm{op}}(x) A \psi_{\mathrm{op}}(x)\, \mathrm{d}x \int \Delta[\psi^*, \Psi]\, \chi[\Psi^*]\, \mathrm{d}\Psi^* = \int_\Omega \psi^*_{\mathrm{op}}(x) A \psi_{\mathrm{op}}(x)\, \mathrm{d}x\, \chi[\psi^*] = \mathscr{A}\chi[\psi^*]\,,$$

we have:

$$\langle \psi'^* | \mathscr{A} | \psi'' \rangle = \Delta[\psi'^*, \psi''] \int_\Omega \psi'^*(x)\, A \psi''(x)\, \mathrm{d}x\,. \tag{57}$$

Since

$$\int \Delta[\psi'^*, \psi] \{ \mathscr{A}_{\psi^*}\, \Delta[\psi^*, \psi''] \}\, \mathrm{d}\psi^* = \mathscr{A}_{\psi'^*} \Delta[\psi'^*, \psi''] = \tag{58}$$

$$= \int_\Omega \psi'^* A \frac{\delta}{\delta\psi'^*(x)}\, \Delta[\psi'^*, \psi'']\, \mathrm{d}x = \int_\Omega \psi'^*(x) A \psi''(x)\, \mathrm{d}x\, \Delta[\psi'^*, \psi'']\,,$$

we have

$$\langle \psi'^* | \mathscr{A} | \psi'' \rangle = \int \Delta[\psi'^*, \psi] \{ \mathscr{A}_{\psi^*}\, \Delta[\psi^*, \psi''] \}\, \mathrm{d}\psi^*, \tag{59}$$

although $\Delta[\psi^*, \Psi]$ is not a normalized functional.

Let us introduce the unitary operator of evolution $\mathcal{U}(t)$ of the χ-formalism, i.e. the operator defined by the equations:

$$i\hbar \frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = \mathcal{H}\mathcal{U}\,, \qquad \mathcal{U}(0) = 1\,. \tag{60}$$

We have:

$$\chi[t;\psi^*] = \mathcal{U}(t)\chi[0;\psi^*]\,. \tag{61}$$

The unitary operator of the motion of Σ is defined by the equations:

$$i\hbar \frac{\mathrm{d}U}{\mathrm{d}t} = HU\,, \qquad U(0) = 1\,. \tag{62}$$

The wave function of Σ at time t corresponding to the wave function $\psi(x)$ at the time 0 is $\psi(t;x)$:

$$\psi(t;x) = U(t)\psi(x)\,, \qquad \psi(0;x) = \psi(x)\,. \tag{63}$$

We have

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \Delta\big[\psi^*(x), \psi'(t;x)\big] = \Delta\big[\psi^*(x), \psi'(t;x)\big]\, i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \int_{\dot{\Omega}} \psi^*(x)\psi'(t;x)\mathrm{d}x = \tag{64}$$

$$= \Delta\big[\psi^*(x), \psi'(t;x)\big] \int_{\dot{\Omega}} \psi^*(x) H \psi'(t;x)\,\mathrm{d}x = \mathcal{H}\Delta\big[\psi^*(x), \psi'(t;x)\big]\,,$$

hence

$$\Delta\big[\psi^*(x), \psi'(t;x)\big] = \mathcal{U}(t)\ \Delta[\psi^*, \psi']\,, \tag{65}$$

so that:

$$\langle \psi'^* | \mathcal{U}(t)\, \psi''\rangle = \Delta\big[\psi'^*(x), \psi''(t;x)\big] = \exp\Big[\int_{\dot{\Omega}} \psi'^*(x)\ U(t)\psi''(x)\,\mathrm{d}x\Big]\,. \tag{66}$$

In the Heisenberg representation of the χ-formalism, the time independent operators $\mathcal{R}$ are replaced by the dependent ones $\mathcal{R}(t)$:

$$\mathcal{R}(t) = \mathcal{U}^{-1}(t)\,\mathcal{R}\mathcal{U}(t)\,, \qquad i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}(t) = \big[\mathcal{R}(t), \mathcal{H}\big]\,. \tag{67}$$

It is easily seen that $\psi_{\mathrm{op}}(t;x)$ has an equation of motion of the same form as the Schrödinger equation (1):

$$i\hbar \frac{\partial}{\partial t}\psi_{\mathrm{op}}(t;x) = H\psi_{\mathrm{op}}(t;x)\,. \tag{68}$$

Hence we have:

$$\psi_{op}(t; x) = U(t)\psi_{op}(x), \qquad \psi^*_{op}(t; x) = U^{-1}(t)\psi^*_{op}(x). \tag{69}$$

The ψ_{op} taken at different times do commute, and the ψ^*_{op} also:

$$[\psi_{op}(t; x), \psi_{op}(t'; x')] = 0, \qquad [\psi^*_{op}(t; x), \psi^*_{op}(t'; x')] = 0. \tag{70}$$

Since

$$\langle \psi'^* | \psi_{op}(t; x) | \psi'' \rangle = U(t)\psi_{op}(x)\, \Delta[\psi'^*, \psi''] = \\ = U(t)\psi''(x)\, \Delta[\psi'^*, \psi''] = \psi''(t; x)\, \Delta[\psi'^*, \psi''], \tag{71}$$

the matrix elements of $\psi_{op}(t; x)$ satisfy the Schrödinger equation (1). We get from (71):

$$\langle \psi'^* | \psi_{op}(t; x) | \psi' \rangle = \psi'(t; x) \exp\left[\int_\Omega |\psi'|^2 \, dx\right]. \tag{72}$$

This equation shows that the wave function Ψ of Σ is the expectation value of $\psi_{op}(t; x)$ taken in the condition of the χ-formalism described by a functional $\Delta[\psi^*, \Psi]$:

$$\Psi(t; x) = (\Delta[\psi^*, \Psi]\, \psi_{op}(t; x) \Delta[\psi^*, \Psi]) \exp\left[-\int_\Omega |\Psi|^2 \, dx\right]. \tag{73}$$

This result follows of course immediately from the fact that $\psi_{op}(x)\Delta[\psi^*, \Psi] = \\ = \Psi(x)\, \Delta[\psi^*, \Psi]$.

Let us consider now the operator L defined by (36). We have

$$L\Delta[\psi^*, \Psi] = \sum_{n=1}^{\infty} \frac{n^{-1}}{n!} \left\{\int_\Omega \psi^*(x)\Psi(x)\, dx\right\}^n = F\left(\int_\Omega \psi^*\Psi \, dx\right), \tag{74}$$

$$F(u) = \int_0^u \frac{e^z - 1}{z}\, dz = Ei(u) - \log u - C. \tag{75}$$

$Ei(u)$ being the exponential integral of u and C the Euler constant. We get from (74) and (37):

$$\langle \psi^* | \mathscr{A}_{nor} | \psi' \rangle = \frac{\int_\Omega \psi^*(x) A \psi'(x)\, dx}{\int_\Omega \psi^*(x)\psi'(x)\, dx} \{\Delta[\psi^*, \psi'] - 1\}, \tag{76}$$

$$\langle \psi^* | \mathscr{A}_{nor} | \psi \rangle = \int_\Omega \psi^*(x) A \psi(x)\, dx \, \frac{\exp\left[\int_\Omega |\psi|^2 \, dx\right] - 1}{\int_\Omega |\psi|^2 \, dx}. \tag{77}$$

3. – The N-representation of the χ-formalism.

We shall now consider the representation of the χ-formalism in which the operators N_λ for a complete orthonormal set of functions φ_λ are diagonal. Instead of a wave functional $\chi[\psi^*]$, we have now a wave function $\chi(N'_1, N'_2, \ldots)$ depending on the eigenvalues of the N_λ. In this representation the inner product is defined by (48). The basic functions of this representation are the $\delta(N', n)$.

$$\delta(N', n) = \prod_\lambda \delta_{N'_\lambda, n_\lambda}, \tag{78}$$

the n being non negative integers, since they are eigenfunctions of the N_λ and:

$$\sum_{N'} \delta^*(N', n')\, \delta(N', n'') = \prod_\lambda \delta_{n'_\lambda, n''_\lambda}, \tag{79a}$$

$$\sum_{n} \delta^*(N', n)\, \delta(N'', n) = \prod_\lambda \delta_{N'_\lambda, N''_\lambda}. \tag{79b}$$

Let us introduce the functionals $\langle N' | \psi \rangle$ and $\langle \psi^* | N' \rangle$:

$$\langle N' | \psi \rangle = \prod_\lambda \left\{ (N'_\lambda !)^{-1/2} \Big(\int_\Omega \varphi^*_\lambda(x) \psi(x)\, \mathrm{d}x \Big)^{N'_\lambda} \right\}, \tag{80}$$

$$\langle \psi^* \ N' \rangle = \{ \langle N' | \psi \rangle \}^*. \tag{81}$$

It is easily seen that:

$$\sum_{N'} \langle \psi^* | N' \rangle \langle N' | \psi' \rangle = \tag{82}$$

$$= \exp\Big[\sum_\lambda \Big(\int_\Omega \varphi^*_\lambda(x) \psi(x)\, \mathrm{d}x \Big)^* \Big(\int_\Omega \varphi^*_\lambda(x) \psi'(x)\, \mathrm{d}x \Big) \Big] = \exp\Big[\int_\Omega \psi^*(x) \psi'(x)\, \mathrm{d}x \Big] = A[\psi^*, \psi'].$$

We have obviously:

$$\langle N' | \psi \rangle = \prod_\lambda \{ (N'_\lambda !)^{-1/2}\, a_\lambda^{N'_\lambda} \}, \tag{83}$$

the a_λ being the complex conjugates of the coefficients in the expansion (42). By taking into account (43) we get:

$$\int \langle N' | \psi \rangle \langle \psi^* | n \rangle\, \mathrm{d}\psi^* = \prod_\lambda \delta_{N'_\lambda, n_\lambda} = \delta(N', n). \tag{84}$$

We get from (27)

$$N_\lambda = \int_\Omega \psi^*_{op}(x)\varphi_\lambda(x)\,dx \int_\Omega \varphi^*_\lambda(x')\psi_{op}(x')\,dx, \tag{85}$$

therefore the $\langle \psi^* \; n \rangle$ are eigenfunctionals of the N_λ corresponding to eigenvalues n_λ:

$$N_\lambda \langle \psi^* | n \rangle = n_\lambda \langle \psi^* | n \rangle . \tag{86}$$

In (86), the n in $\langle \psi^* | n \rangle$ are treated as indices, $\langle \psi^* | n \rangle$ being considered as a wave functional in the ψ^*-representation.

In the application of the χ-formalism to the second quantization, the wave functional $\chi[\psi^*]$ of a state of the quantized ψ-field is related to the wave function $\chi(N')$ of the same state by the equation:

$$\chi[\psi^*] = \sum_{N'} \langle \psi^* | N' \rangle \chi(N') . \tag{87}$$

The geometrical meaning of (87) becomes clear by taking into account that $\langle \psi^* \; N' \rangle$ is an eigenfunctional of the N_λ: $\chi[\psi^*]$ is the component of the vector representing the state of the ψ-field in the direction of the eigenvector of the $\psi^*_{op}(x)$ corresponding to the eigenvalues $\psi^*(x)$, $\langle \psi^* | N' \rangle$ the projection on the direction on that eigenvector of a normalized eigenvector of the N_λ corresponding to the eigenvalues N'_λ and $\chi(N')$ the projection of the vector of the state on the direction of this eigenvector of the N'. It follows from (87) and (84) that:

$$\chi(N') = \int \langle N' | \psi \rangle \chi[\psi^*]\, d\psi^* . \tag{88}$$

This equation shows that $\langle N' | \Psi \rangle$ must be an eigenfunction of the operators $\psi^{(N)}_{op}(x)$ which correspond to the $\psi_{op}(x)$ in the N-representation:

$$\psi^{(N)}_{op}(x) \langle N' | \Psi \rangle = \Psi(x) \langle N' | \Psi \rangle . \tag{89}$$

Equation (89) can be easily checked by using the well known expression of $\psi^{(N)}_{op}(x)$:

$$\psi^{(N)}_{op}(x) = \sum_\lambda \varphi_\lambda(x) \sqrt{N_\lambda + 1}\, \exp\left[\frac{\partial}{\partial N_\lambda}\right] . \tag{90}$$

It results from (87) and (81) that:

$$\chi^*[\psi] = \sum_{N'} \chi^*(N') \langle N' | \psi \rangle . \tag{91}$$

Hence

$$\int \chi^*[\psi]\chi'[\psi^*]\,\mathrm{d}\psi^* = \sum_{N',N''} \chi^*(N')\chi'(N'') \int \langle N'|\psi\rangle\langle\psi^*|N''\rangle\,\mathrm{d}\psi^* = \sum_{N'} \chi^*(N')\chi'(N')\,, \tag{92}$$

and thus the equivalence of the inner product definitions (6) and (48) is proven.

Let us consider the spectral decomposition of the hermitian operator A with a discrete spectrum:

$$A = \sum_{A'} \sum_{\alpha} A' p_{A'}^{(\alpha)}\,, \tag{93}$$

$p_{A'}^{(\alpha)}$ denoting the projection operator on the direction of the normalized eigenfunction $\varphi_{A'}^{(\alpha)}(\alpha)$. It follows from (93) that the corresponding χ-operator $\mathscr{A}$ can be written as follows

$$\mathscr{A} = \sum_{A'} \sum_{\alpha=1}^{s_{A'}} A' N_{A'}^{(\alpha)}\,, \tag{94}$$

$s_{A'}$ being the number of linearly independent eigenfunctions of the eigenvalue A':

$$N_{A'}^{(\alpha)} = \int_{\Omega} \psi_{\mathrm{op}}^*(x) p_{A'}^{(\alpha)} \psi_{\mathrm{op}}(x)\,\mathrm{d}x\,. \tag{95}$$

The eigenvalues of the $N_{A'}^{(\alpha)}$ are the non negative integers. Therefore the eigenvalues of $\mathscr{A}$ are all the linear combinations $\sum_{A',\alpha} n_{A'}^{(\alpha)} A'$ with integer non negative coefficients $n_{A'}^{(\alpha)}$. It follows from (94), (37) and (36) that:

$$\mathscr{A}_{\mathrm{nor}} = \sum_{A',\alpha} A' \sum_{n=1}^{\infty} n^{-1} P_n N_{A'}^{(\alpha)}\,. \tag{96}$$

The eigenvalues of $P_n N_{A'}^{(\alpha)}$ are $0, 1, 2, \ldots, n$. Therefore the eigenvalues or $\mathscr{A}_{\mathrm{nor}}$ are linear combinations $\sum_{A',\alpha} r_{A'}^{(\alpha)} A'$, the $r_{A'}^{(\alpha)}$ being non negative rational numbers smaller than 1. The $r_{A'}^{(\alpha)}$ cannot be chosen arbitrarily, because $P_n P_{n'} = 0$, for $n \neq n'$. We can take

$$r_{A'}^{(\alpha)} = l_{A'}^{(\alpha)} \Big/ \sum_{A'',\alpha'} l_{A''}^{(\alpha')} \quad \text{when} \quad \sum_{A',\alpha} l_{A'}^{(\alpha)} \neq 0\,; \qquad r_{A'}^{(\alpha)} = 0 \quad \text{when} \quad \sum_{A',\alpha} l_{A'}^{(\alpha)} = 0\,, \tag{97}$$

the $l_{A'}^{(\alpha)}$ being non negative integers. *The eigenvalues of $\mathscr{A}_{\mathrm{nor}}$ cover densely the interval $A'_{\min} - A'_{\max}$, the only eigenvalue of $\mathscr{A}_{\mathrm{nor}}$ lying outside that interval being possibly 0, in case 0 does not belong to that interval.*

It is easily seen that:

$$\int \chi^*[\psi]\, \Delta[\psi^*, \psi']\, \mathrm{d}\psi^* = \chi^*[\psi']\,. \tag{98}$$

It follows from (88) and (98) that $\langle N'|\Psi\rangle$ is the wave function of the N-representation which corresponds to $\Delta[\psi^*, \Psi]$:

$$\langle N'|\Psi\rangle = \int \langle N'|\psi\rangle\, \Delta[\psi^*, \Psi]\, \mathrm{d}\psi^*\,, \tag{99}$$

$\langle \psi^*|n\rangle$ is the wave functional which corresponds to $\delta(N', n)$:

$$\langle \psi^*|n\rangle = \sum_{N'} \langle \psi^*|N'\rangle\, \delta(N', n)\,. \tag{100}$$

We shall denote by $\chi[N'; \Psi]$ the wave function which corresponds to the functional (2):

$$\chi[N'; \Psi] = \int \langle N'|\psi\rangle \left\{ \int_\Omega \psi^*(x)\Psi(x)\, \mathrm{d}x \right\} \mathrm{d}\psi^* = \tag{101}$$

$$= \sum_\lambda \int_\Omega \varphi_\lambda^*(x)\Psi(x)\, \mathrm{d}x\, \delta_{N'_\lambda, 1} \prod_{\mu \neq \lambda} \delta_{N'_\mu, 0}\,.$$

By taking into account that

$$\psi_{\mathrm{op}}^{*(N)}(x) = \sum_\lambda \varphi_\lambda^*(x) \sqrt{N_\lambda}\, \exp\left[-\frac{\partial}{\partial N_\lambda} \right], \tag{102}$$

we get:

$$\chi[N'; \Psi] = \int_\Omega \psi_{\mathrm{op}}^{*(N)}(x)\Psi(x)\, \mathrm{d}x\, \delta(N', 0)\,. \tag{103}$$

We shall denote by $\chi[N'; \Psi_n(x_1, \ldots, x_n)]$ the N' wave function corresponding to the homogeneous functional (16):

$$\chi[N'; \Psi_n(x_1, \ldots, x_n)] = \tag{104}$$

$$= \frac{1}{\sqrt{n!}} \int \langle N'\ \psi\rangle \left\{ \int_\Omega \Psi_n(x_1, \ldots, x_n)\psi^*(x_1) \ldots \psi^*(x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n \right\} \mathrm{d}\psi^*\,.$$

The ψ^* integration can be easily performed by applying (45) to the homogeneous functional and using the expression (83) of $\langle N'|\psi\rangle$:

$$\chi[N'; \Psi_n(x_1, \ldots, x_n)] = \frac{1}{\sqrt{n!}} \sum_{\lambda_1, \ldots, \lambda_n} \int_\Omega \Psi_n(x_1, \ldots, x_n)\varphi_{\lambda_1}^*(x_1) \ldots \varphi_{\lambda_n}^*(x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n \cdot \tag{105}$$

$$\cdot \sqrt{s_1!\, s_2!\, \ldots\, s_r!}\ \delta_{N'_{\mu_1}, s_1}\ \delta_{N'_{\mu_2}, s_2} \ldots \delta_{N'_{\mu_r}, s_r} \prod_{\lambda \neq \mu} \delta_{N'_\lambda, 0}\,,$$

$(\mu_1, \mu_2, ..., \mu_n)$ denote the distinct numbers λ in the sequence $\lambda_1, ..., \lambda_n$ and the s the frequency in that sequence of the corresponding μ.

4. – The general occupation number operators N_M.

Until now we considered only operators N_φ for a state φ of Σ and the operator N_{op}. Let M be any closed linear manifold of the Hilbert space of the ψ and p_M the corresponding projection operator. p_M transforms any vector ψ into its projection on M. The occupation number operator N_M of M is defined as the χ-operator corresponding to p_M:

$$N_M = \int_\Omega \psi^*_{op}(x) p_M \psi_{op}(x)\, dx . \tag{106}$$

Let the $\psi^{(\varrho)}_M(x)$ be a complete orthonormal set of vectors for M. Since the projection of a vector on M is the sum of its projections on the directions of the $\psi^{(\varrho)}_M$, the projection operator p_M is the sum of the projection operators $p^{(\varrho)}_M$ on the directions of the vectors $\psi^{(\varrho)}_M$:

$$p_M = \sum_\varrho p^{(\varrho)}_M . \tag{107}$$

Hence

$$N_M = \sum_\varrho N^{(\varrho)}_M , \qquad N^{(\varrho)}_M = N_{\psi^{(\varrho)}_M} : , \tag{108}$$

so that the occupation number operator N_M of the manifold M is the sum of the occupation number operators of any complete orthonormal set of vectors of M. N_{op} is simply the occupation number operator of the entire ψ Hilbert space.

To each discrete eigenvalue A' of the operator of a physical quantity (A) corresponds a closed linear manifold $M_{A'}$, whose basic vectors are those corresponding to a complete orthonormal set $\psi^{(x)}_{A'}$ for the eigenvalue A'. We shall denote the projection operator of $M_{A'}$ by $p_{A'}$ and the corresponding occupation number operator by $N_{A'}$. Equations (93) and (94) can be written as follows:

$$A = \sum_{A'} A' p_{A'} , \tag{109}$$

$$\mathcal{A} = \sum_{A'} A' N_{A'} . \tag{110}$$

The general form of the spectral decomposition of a hermitian operator is:

$$A = \int_{-\infty}^{+\infty} \lambda \, \mathrm{d}E_A(\lambda) \,. \tag{111}$$

The $E_A(\lambda)$ are projection operators on manifolds $\mathcal{M}_A(\lambda)$. They satisfy the following equations:

$$E_A(-\infty) = 0 \,, \qquad E_A(\infty) = 1 \,, \qquad E_A(\lambda + 0) = E_A(\lambda) \,, \tag{112}$$

$$E_A(\lambda')E_A(\lambda'') = E_A(\lambda'')E_A(\lambda') = E_A(\lambda') \,, \qquad (\lambda' \leqslant \lambda'') \,. \tag{113}$$

In the case of a pure point spectrum we can take:

$$E_A(\lambda) = \sum_{A' \leqslant \lambda} p_{A'} = \sum_{A'} \theta(\lambda - A') p_{A'} = \theta(\lambda - A) \,, \tag{114}$$

$$\theta(\lambda) = \begin{cases} 1 & \text{for} \quad \lambda \geqslant 0 \,, \\ 0 & \text{for} \quad \lambda < 0 \,. \end{cases} \tag{115}$$

The occupation number operator of $\mathcal{M}_A(\lambda)$ will be denoted by $N_A(\lambda)$:

$$N_A(\lambda) = \int_{\Omega} \psi^*_{o\mathfrak{d}}(x) E_A(\lambda) \psi_{o\mathfrak{d}}(x) \, \mathrm{d}x \,. \tag{116}$$

Equation (110) can be generalized as follows:

$$\mathcal{A} = \int_{-\infty}^{+\infty} \lambda \, \mathrm{d}N_A(\lambda) \,. \tag{117}$$

$E_A(\lambda'') - E_A(\lambda')$, $\lambda' < \lambda''$, is the projection operator of a closed linear manifold $\mathcal{M}_A(\lambda', \lambda'')$. In the case of a discrete spectrum $\mathcal{M}_A(\lambda', \lambda'')$ is the linear manifold whose basic vectors are the $\psi^{(\alpha)}_{A'}$ for all the eigenvalues of A in the semi-closed interval $(\lambda', \lambda'']$, i.e. for $\lambda' < A' \leqslant \lambda''$.

A family of projection operators satisfying the conditions (112) and (113) is called a spectral series. To any spectral series corresponds a set of operators $N(\lambda)$ and there is a representation in which all the $N(\lambda)$ are diagonal. In this representation the wave function is actually a wave functional $\chi[N'(\lambda)]$, depending on a non decreasing function $N'(\lambda)$ with non negative integer values.

p_M is a non normalized density operator:

$$\langle x' | p_M | x'' \rangle = \sum_{\varrho} \langle x' | p^{(\varrho)}_M | x'' \rangle = \sum_{\varrho} \psi^{(\varrho)}_M(x') \psi^{(\varrho)}_M{}^*(x'') \,, \tag{118}$$

$$\text{Trace } p_M = \sum_{\varrho} \delta_{\varrho,\varrho} \,. \tag{119}$$

The states $\psi_M^{(\varrho)}$ are equally probable in the mixture described by p_M. By assigning different weights W_ϱ to the basic vectors of M we get a general density operator R_M attached to the manifold M:

$$R_M = \sum_\varrho W_\varrho p_M^{(\varrho)}\,, \qquad W_\varrho \geqslant 0\,. \tag{120}$$

There is no restriction of generality in considering only density operators attached to the entire Hilbert space, since it suffices to give weights 0 to the basic vectors of the manifold orthogonal to M, in case M is not empty. Thus we shall take

$$R = \sum_\lambda W_\lambda p_\lambda\,, \qquad W_\lambda \geqslant 0 \tag{121}$$

the p_λ being the projection operators of the functions of a complete orthonormal set φ_λ.

The equation of evolution of the density operator $R(t)$ describing a mixed state of motion of Σ is the von Neumann equation (12). Let us introduce the χ-operator $N_{R(t)}$:

$$N_{R(t)} = \int_\Omega \psi_{\mathrm{op}}^*(x) R(t) \psi_{\mathrm{op}}(x)\,\mathrm{d}x\,. \tag{122}$$

We have

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} N_{R(t)} = \int_\Omega \psi_{\mathrm{op}}^*(x)\,[H,\, R(t)]\,\psi_{\mathrm{op}}(x)\,\mathrm{d}x = [\mathcal{H},\, N_{R(t)}]\,, \tag{123}$$

as a consequence of the equation:

$$\int_\Omega \psi_{\mathrm{op}}^*(x)[A,\, B]\psi_{\mathrm{op}}(x)\,\mathrm{d}x = [\mathcal{A},\, \mathcal{B}]\,. \tag{124}$$

Equation (124) follows immediately from the following equation

$$\begin{aligned}\int_\Omega \psi_{\mathrm{op}}^*(x) A_x \psi_{\mathrm{op}}(x)\,\mathrm{d}x \int_\Omega \psi_{\mathrm{op}}^*(x') B_{x'} \psi_{\mathrm{op}}(x')\,\mathrm{d}x' = \\ = \int_\Omega \psi_{\mathrm{op}}^*(x)\psi_{\mathrm{op}}^*(x') A_x B_{x'} \psi_{\mathrm{op}}(x')\psi_{\mathrm{op}}(x)\,\mathrm{d}x\,\mathrm{d}x' + \int_\Omega \psi_{\mathrm{op}}^*(x) A_x B_x \psi_{\mathrm{op}}(x)\,\mathrm{d}x\,,\end{aligned} \tag{125}$$

the index x denoting the variable on which the operator acts, by taking into account that:

$$[A_x,\, B_{x'}] = 0\,, \tag{126}$$

$N_{R(t)}$ is a density operator of the χ-formalism, because it is hermitian, with positive eigenvalues, and its equation of evolution (123) has the form of a von Neumann equation of the χ-formalism:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{R}(t) = [\mathcal{H}, \mathcal{R}(t)] = \mathcal{H}\mathcal{R}(t) - \{\mathcal{H}\mathcal{R}(t)\}^* . \tag{127}$$

We get from (127):

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle = \langle \psi'^* | \mathcal{H}\mathcal{R}(t) | \psi'' \rangle - \langle \psi''^* | \mathcal{H}\mathcal{R}(t) | \psi' \rangle^* . \tag{128}$$

Since

$$\langle \psi'^* | \mathcal{H}\mathcal{R}(t) | \psi'' \rangle = \{\mathcal{H}\mathcal{R}(t)\}_{\psi'^*} \Delta[\psi'^*, \psi''] = \mathcal{H}_{\psi'^*} \langle \psi'^* | \mathcal{R}(t) \ \psi'' \rangle = \tag{129}$$

$$= \int_{\Omega} \psi'^*(x) H \frac{\delta}{\delta \psi'^*(x)} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle \, \mathrm{d}x ,$$

we have:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle = \tag{130}$$

$$= \int_{\Omega} \left\{ \psi'^*(x) H \frac{\delta}{\delta \psi'^*(x)} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle - \left\{ \frac{\delta}{\delta \psi''(x)} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle \right\} H \psi''(x) \right\} \mathrm{d}x .$$

The density operators $\mathcal{R}$ describe mixtures of states of the quantized ψ-field. They can also be applied to the one system χ-formalism as will be shown in a following paper.

5. – The R density operators of the χ-formalism.

It is easily seen that the density operator $R(t)$ defined by (11) can be expressed as follows:

$$\langle x' | R(t) | x'' \rangle = (\chi[t; \psi^*], \psi^*_{\mathrm{op}}(x'') \psi_{\mathrm{op}}(x') L \chi[t; \psi^*]) . \tag{131}$$

It results from (61) that

$$\langle x' | R(t) | x'' \rangle = (\chi[0; \psi^*], \mathcal{U}^{-1}(t) \psi^*_{\mathrm{op}}(x'') \psi_{\mathrm{op}}(x') \mathcal{U}(t) L \chi[0; \psi^*]) , \tag{132}$$

since

$$[L, \mathcal{U}(t)] = 0 , \tag{133}$$

as a consequence of the commutability of $\mathcal{H}$ and the projection operators P_n of N_{op}. By taking into account (67) and (69) we get from (132):

$$\langle x'|R(t)|x''\rangle = (\chi[0;\psi^*],\ \psi^*_{op}(t;x'')\psi_{op}(t;x')L\chi[0;\psi^*]) = \tag{134}$$
$$= (\psi_{op}(t;x'')\chi[0;\psi^*],\ \psi_{op}(t;x')L\chi[0;\psi^*]) =$$
$$= (U(t)\psi_{op}(x'')\chi[0;\psi^*],\ U(t)\psi_{op}(x')L\chi[0;\psi^*])\ .$$

It follows from (134) and (62) that

$$i\hbar\frac{\partial}{\partial t}\langle x'|R(t)|x''\rangle = (-H_{x''}\psi_{op}(t;x'')\chi[0;\psi^*],\ \psi_{op}(t;x')L\chi[0;\psi^*]) + \tag{135}$$
$$+(\psi_{op}(t;x'')\chi[0;\psi^*],\ H_{x'}\psi_{op}(t;x')L\chi[0;\psi^*]) =$$
$$=(\chi[0;\psi^*],\ \psi^*_{op}(t;x'')H_{x'}\psi_{op}(t;x')L\chi[0;\psi^*])-$$
$$-(L\psi^*_{op}(t;x')H_{x''}\psi_{op}(t;x'')\chi[0;\psi^*],\ \chi[0;\psi^*])\,,$$

and we get finally:

$$i\hbar\frac{\partial}{\partial t}\langle x'|R(t)|x''\rangle = H_{x'}\langle x'|R(t)|x''\rangle - \{H_{x''}\langle x''|R(t)|x'\rangle\}^*\ . \tag{136}$$

Equation (136) is equivalent to (12). $R(t)$ is a density operator, because it is hermitian and has only positive discrete eigenvalues, since Trace $R(t) < \infty$ for a normalizable χ, and

$$\int_\Omega \Psi^*(x)R(t)\Psi(x)\,dx = \int_\Omega \Psi^*(x')\langle x'|R(t)|x''\rangle\Psi(x'')\,dx'\,dx'' = \tag{137}$$
$$= \sum_{n=1}^{\infty}\int_\Omega dx_1\ldots dx_{n-1}\left|\int_\Omega \Psi^*(x)\,\Psi_n(t;x,x_1,\ldots,x_{n-1})\,dx\right|^2 \geqslant 0\,,$$

so that $R(t)$ is a satisfactory density operator for a system Σ.

It is interesting to remark that we can get any density operator of the form (121) from a quadratic functional. Indeed, by taking

$$\Psi_2(x_1,x_2) = \sum_\lambda \sqrt{W_\lambda}\varphi_\lambda(x_1)\varphi_\lambda(x_2)\,, \qquad \Psi_n = 0 \quad \text{for} \quad n \neq 2\,, \tag{138}$$

we get:

$$\langle x'|R|x''\rangle = \sum_\lambda W_\lambda\varphi_\lambda(x')\varphi^*_\lambda(x'')\,. \tag{139}$$

We have

$$\text{Trace } R = (\chi, \chi) - |\Psi_0|^2 , \tag{140}$$

thereby R will in general not be a normalized density operator, for a normalized χ, but it will be normalized in the important case of $\Psi_0 = 0$.

It is well known that the expectation value of a physical quantity (A) in the mixed state of Σ described by the normalized density operator R is the Trace of AR. We have

$$\begin{aligned}(\chi, \mathscr{A}_{\text{nor}}\chi) &= \Big(\chi, \int_{\Omega} \psi^*_{\text{op}}(x) A \psi_{\text{op}}(x)\, \mathrm{d}x\, L\chi\Big) = \\ &= \Big(\chi, \int_{\Omega} \psi^*_{\text{op}}(x') \langle x' | A | x'' \rangle \psi_{\text{op}}(x'')\, \mathrm{d}x'\, \mathrm{d}x''\, L\chi\Big) = \\ &= \int_{\Omega} \mathrm{d}x'\, \mathrm{d}x'' \langle x' | A | x'' \rangle (\chi,\ \psi^*_{\text{op}}(x') \psi_{\text{op}}(x'') L\chi) = \\ &= \int_{\Omega} \mathrm{d}x'\ \mathrm{d}x'' \langle x' | A | x'' \rangle \langle x'' | R | x' \rangle ,\end{aligned} \tag{141}$$

hence:

$$(\chi, \mathscr{A}_{\text{nor}}\chi) = \text{Trace } (AR) . \tag{142}$$

Thereby the expectation value of (A) in the mixed state described by R is given by the χ expectation value $(\chi, \mathscr{A}_{\text{nor}}\chi)$ divided by $1 - |\Psi_0|^2$, for a normalized χ.

Let us consider now a wave functional $\chi(t)$ in whose Volterra expansion (4) there are terms with $n \geqslant r$, r being a positive integer. With that functional $\chi(t)$ we can build a r-system density operator R_r:

$$\begin{aligned}\langle x'_1, \ldots, x'_r | R_r(t) | x''_1, \ldots, x''_r \rangle &= \\ &= (\chi[t; \psi^*],\ \psi^*_{\text{op}}(x''_1) \ldots \psi^*_{\text{op}}(x''_r) \psi_{\text{op}}(x'_r) \ldots \psi_{\text{op}}(x'_1) \chi[t; \psi^*]) = \\ &= (\chi[0; \psi^*],\ \psi^*_{\text{op}}(t; x''_1) \ldots \psi^*_{\text{op}}(t; x''_r) \psi_{\text{op}}(t; x'_r) \ldots \psi_{\text{op}}(t; x'_1) \chi[0; \psi^*]) .\end{aligned} \tag{143}$$

We may write

$$\begin{aligned}&\langle x'_1, \ldots, x'_r | R_r(t) | x''_1, \ldots, x''_r \rangle = \\ &= (\psi_{\text{op}}(t; x''_1) \ldots \psi_{\text{op}}(t; x''_r) \chi[0; \psi^*],\ \psi_{\text{op}}(t; x'_1) \ldots \psi_{\text{op}}(t; x'_r) \chi[0; \psi^*]) = \\ &= (U(t)_{x'_1} \ldots U(t)_{x'_r} \psi_{\text{op}}(x''_1) \ldots \psi_{\text{op}}(x''_r) \chi[0; \psi^*],\ U(t)_{x'_1} \ldots U(t)_{x'_r} \psi_{\text{op}}(x'_1) \ldots \psi_{\text{op}}(x'_r) \chi[0; \psi^*]) ,\end{aligned} \tag{144}$$

therefore:

$$(145)\qquad i\hbar\frac{\partial}{\partial t}\langle x'_1,\ldots,x'_r|R_r(t)|x''_1,\ldots,x''_r\rangle = \\ = (H_{x'_1}+\ldots+H_{x'_r})\langle x'_1,\ldots,x'_1|R_r(t)|x''_1,\ldots,x'_r\rangle - \\ -\{(H_{x''_1}+\ldots+H_{x''_r})\langle x''_1,\ldots,x''_r|R_r(t)|x'_1,\ldots,x'_r\rangle\}^* .$$

This equation shows that the hermitian operator $R_r(t)$ is a solution of the von Neumann equation for an assembly of r non interacting systems Σ. By a reasoning similar to that applied to $R(t)$, it is seen that the eigenvalues of $R_r(t)$ are non negative. $R_r(t)$ is therefore a density operator for an assembly of r non interacting systems Σ.

It is interesting to remark that with a functional $\chi[t,\psi^*]$ it is sometimes possible to build a solution of the Schrödinger equation (1):

$$(146)\qquad \Psi(t;x) = (\chi[t;\psi^*],\psi_{\text{op}}(x)\chi[t;\psi^*]) = (\chi[0;\psi^*],\psi_{\text{op}}(t;x)\chi[0;\psi^*]) .$$

Any density operator R of the system Σ can be obtained in an infinite number of different ways from functionals χ. In order to obtain the density operator of (139) by means of a homogeneous functional of the type (16), it suffices to take

$$(147)\qquad \Psi_n(x_1,\ldots,x_n) = \sum_\lambda \sqrt{W_\lambda}\,\varphi_\lambda(x_1)\ldots\varphi_\lambda(x_n) ,$$

since the operator $R^{(n)}$ associated to a χ_n has the matrix elements:

$$(148)\qquad \langle x'|R^{(n)}|x''\rangle = \int_\Omega \Psi_n(x',x_1,\ldots,x_{n-1})\Psi_n^*(x'',x_1,\ldots,x_{n-1})\,\mathrm{d}x\ldots\mathrm{d}x_{n-1} .$$

In particular the homogeneous functional χ_n whose Ψ_n is

$$(149)\qquad \Psi_n(x_1,\ldots,x_n) = \varphi(x_1)\ldots\varphi(x_n) ,$$

corresponds to a density operator describing the pure state of wave function φ.

The density operator corresponding to the functional whose Ψ_n are

$$(150)\qquad \Psi_0 = 0 , \qquad \Psi_n(x_1,\ldots,x_n) = \sqrt{W_n}\varphi_n(x_1)\ldots\varphi_n(x_n) \qquad \text{for} \quad n \geqslant 1 ,$$

is the operator (139). It is interesting to remark that each term in the expansion of the type (4) of the functional defined by (150) has a density matrix describing a pure state.

In the particular case of normalized functionals χ with $\Psi_0 = 0$, the corresponding operators R have trace 1. Thereby Trace (AR) is the expectation value of the quantity (A) in the mixed state described by R. By taking into account (142), we see that the field expectation value $(\chi, \mathscr{A}_{\text{nor}}\chi)$ of $\mathscr{A}_{\text{nor}}$ coincides with that expectation value of (A). Since the nor χ-operator which corresponds to the projection operator $p_{A'}$, is $LN_{A'}$, as shown by (106), the probability of obtaining the value A' in a measurement of (A) in the mixed state described by the R corresponding to χ is $(\chi, LN_{A'}\chi)$:

$$\text{Trace}\,(p_{A'}R) = (\chi, LN_{A'}\chi)\,, \qquad ((\chi, \chi) = 1,\ \Psi_0 = 0)\,. \tag{151}$$

6. – Description of mixtures of states of Σ by functionals χ.

The results of section 5 show that it is possible to describe the mixtures of states of Σ, as well as the pure states, by means of functionals χ. It suffices to take the R operators attached to χ as describing the mixtures and pure states of Σ. In this description of the states of Σ, it is convenient to consider only normalized functionals with $\Psi_0 = 0$. With this choice of the χ the physical quantity will be described by the operator $\mathscr{A}_{\text{nor}}$, provided the spectral decomposition of $\mathscr{A}_{\text{nor}}$ be replaced by the following decomposition:

$$\mathscr{A}_{\text{nor}} = \sum_{A'} A' LN_{A'}\,. \tag{152}$$

Equation (152) is a consequence of (110). Thus the probability of obtaining the value A' of the quantity (A) in the mixed state described by χ is given by the expectation value $(\chi, LN_{A'}\chi)$, as was shown in section 5. This corresponds to the ordinary rule for the computation of the probability of obtaining the value A' of (A) in the pure state of Σ described by the normalized wave function Ψ, that probability being the expectation value $\int_{\Omega} \Psi^* p_{A'} \Psi \,\mathrm{d}x$ of $p_{A'}$. Now both the pure and mixed states of Σ are described by functionals χ i.e. correspond to pure states of the χ-formalism, and the projection operator $p_{A'}$ is replaced by $LN_{A'}$.

The above interpretation of the χ-formalism by means of the functionals R corresponds to a probability distribution in the ψ-space, in which probabilities are assigned to linear manifolds. Indeed, let us assign to the linear manifolds M the probabilities $\mathscr{P}[M]$:

$$\mathscr{P}[M] = (\chi, LN_M\chi)\,, \qquad ((\chi, \chi) = 1, \Psi_0 = 0)\,. \tag{153}$$

Let M_1 and M_2 be two orthogonal linear manifolds. Since

$$p_{M_1+M_2} = p_{M_1} + p_{M_2}\,, \tag{154}$$

we have:

$$N_{M_1+M_2} = N_{M_1} + N_{M_2} . \tag{155}$$

This equation shows that two orthogonal manifolds correspond to exclusive events:

$$\mathcal{P}[M_1 + M_2] = \mathcal{P}[M_1] + \mathcal{P}[M_2] . \tag{156}$$

The probability of the total Hilbert space S is 1

$$\mathcal{P}[S] = (\chi, LN_{op}\chi) = 1 , \tag{157}$$

since $LN_{op}\chi = \chi$, because $\Psi_0 = 0$. We get the above rule of interpretation by assuming that the probability of obtaining a value A' of the quantity (A) in a measurement performed in the mixed state described by χ coincides with the probability of the manifold $M_{A'}$ in the probability distribution associated to χ.

In the case of the general spectrum we must introduce the linear manifolds $\mathfrak{M}_A(\lambda', \lambda'')$ of section 4. The probability of $\mathfrak{M}_A(\lambda', \lambda'')$

$$\mathcal{P}[\mathfrak{M}_A(\lambda', \lambda'')] = (\chi, L\{N_A(\lambda'') - N_A(\lambda')\}\chi) , \tag{158}$$

coincides with that of obtaining a value λ of (A) in the semi-closed interval $(\lambda', \lambda'']$, $\lambda' < \lambda \leqslant \lambda''$.

Let us consider the particular case of a linear functional (2). The density operator R describes now a pure state, as shown by (13), precisely the pure state Ψ. The probability of the ray defined by the normalized vector φ is $|\chi[\varphi^*]|^2$:

$$(\chi, LN_\varphi\chi) = (\chi, N_\varphi\chi) = \left| \int_\Omega \Psi(x)\varphi^*(x)\,dx \right|^2 = |\chi[\varphi^*]|^2 . \tag{159}$$

Therefore in the case of a linear functional the probability of the ray φ coincides with the probability of the normalized function φ according to the probability distribution of the wave functions corresponding to $\chi[\psi^*]$. In this case $|\chi[\psi^*]|^2$ does not depend on the choice of the constant phase factor in the normalization of φ.

We shall now introduce a functional $\bar{\Delta}[\psi^*, \Psi]$ which differs slightly from $\Delta[\psi^*, \Psi]$

$$\bar{\Delta}[\psi^*, \Psi] = (1 - P_0)\,\Delta[\psi^*, \Psi] = \Delta[\psi^*, \Psi] - 1 , \tag{160}$$

$$\bar{\Delta}[\psi^*, \psi'] = \langle \psi^* | 1 - P_0 | \psi' \rangle = \left\langle \psi^* \middle| \sum_{n=1}^{\infty} P_n \middle| \psi' \right\rangle . \tag{161}$$

We have obviously

$$\int \bar{\Delta}[\psi^*, \psi']\chi[\psi'^*]\,\mathrm{d}\psi'^* = \chi[\psi^*] - \Psi_0\,, \tag{162}$$

in particular

$$(\bar{\Delta}[\psi^*, \Psi], \bar{\Delta}[\psi^*, \Psi]) = \bar{\Delta}[\Psi, \Psi] = \exp\left[\int_\Omega |\Psi|^2\,\mathrm{d}x\right] - 1\,. \tag{163}$$

It is easily seen that:

$$\psi_{\mathrm{op}}(x)\,\bar{\Delta}[\psi^*, \Psi] = \psi_{\mathrm{op}}(x)\,\Delta[\psi^*, \Psi] = \Psi(x)\{\bar{\Delta}[\psi^*, \Psi] + 1\}\,, \tag{164}$$

$$\mathscr{A}\bar{\Delta}[\psi^*, \Psi] = \mathscr{A}\Delta[\psi^*, \Psi] = \Delta[\psi^*, \Psi]\int_\Omega \psi^* A\Psi\,\mathrm{d}x\,, \tag{165}$$

$$L\bar{\Delta}[\psi^*, \Psi] = L\Delta[\psi^*, \Psi], \tag{166}$$

$$\mathscr{A}_{\mathrm{nor}}\bar{\Delta}[\psi^*, \Psi] = \mathscr{A}L\bar{\Delta}[\psi^*, \Psi] = \mathscr{A}_{\mathrm{nor}}\Delta[\psi^*, \Psi]\,. \tag{167}$$

It follows from (167) and (76) that:

$$\mathscr{A}_{\mathrm{nor}}\bar{\Delta}[\psi^*, \psi'] = \langle\psi^*|\mathscr{A}_{\mathrm{nor}}|\psi'\rangle = \bar{\Delta}[\psi^*, \psi']\frac{\int_\Omega \psi^* A\psi'\,\mathrm{d}x}{\int_\Omega \psi^*\psi'\,\mathrm{d}x}\,, \tag{168}$$

In particular we have:

$$\mathscr{A}_{\mathrm{nor}}\bar{\Delta}[\psi^*, \psi_{A'}] = A'\,\Delta[\psi^*, \psi_{A'}]\,. \tag{169}$$

Therefore $\bar{\Delta}[\psi^*, \Psi_{A'}]$ is an eigenfunctional of $\mathscr{A}_{\mathrm{nor}}$ corresponding to the eigenvalue A'.

It results from (168), with a normalized Ψ, that

$$\frac{1}{e-1}(\bar{\Delta}[\psi^*, \Psi], \mathscr{A}_{\mathrm{nor}}\bar{\Delta}[\psi^*, \Psi]) = \frac{1}{e-1}\{\mathscr{A}_{\mathrm{nor}}\Delta[\psi^*, \Psi]\}_{(\psi=\Psi)} = \int_\Omega \Psi^* A\Psi\,\mathrm{d}x, \tag{170}$$

and by taking $A = p_M$ we get:

$$\frac{1}{e-1}(\bar{\Delta}[\psi^*, \Psi], LN_M\bar{\Delta}[\psi^*, \Psi]) = \int_\Omega \Psi^* p_M\Psi\,\mathrm{d}x\,. \tag{171}$$

Therefore the probability distributions corresponding to $(1/\sqrt{e-1})\bar{\Delta}[\psi^*, \Psi]$

and to $\int_{\Omega} \psi^* \Psi \, \mathrm{d}x$ are the same. It is interesting to notice that $\int_{\Omega} \psi^* \psi_{A'} \, \mathrm{d}x$ is also an eigenfunctional of $\mathscr{A}_{\mathrm{nor}}$ corresponding to the eigenvalue A':

$$\mathscr{A}_{\mathrm{nor}} \int_{\Omega} \psi^* \psi'_{A'} \, \mathrm{d}x = \mathscr{A} \int_{\Omega} \psi^* \psi_{A'} \, \mathrm{d}x = \int_{\Omega} \psi^* A \psi_{A'} \, \mathrm{d}x = A' \int_{\Omega} \psi^* \psi_{A'} \, \mathrm{d}x \,. \tag{172}$$

The correspondence between the normalized wave functions Ψ of Σ and the functionals $\bar{\Delta}[\psi^*, \Psi]$ has the remarkable property of conserving the orthogonality. Indeed we have:

$$(\bar{\Delta}[\psi^*, \Psi_1], \Delta[\psi^*, \Psi_2]) = \bar{\Delta}[\Psi_1^*, \Psi_2] = \exp\Big[\int_{\Omega} \Psi_1^* \Psi_2 \, \mathrm{d}x\Big] - 1 \,. \tag{173}$$

In the correspondence between wave functions $\Psi(x)$ and linear functionals $\int_{\Omega} \psi^* \Psi \, \mathrm{d}x$ not only the orthogonality but the inner products themselves are conserved, as shown by (6*a*). $\int_{\Omega} \psi^* \Psi \, \mathrm{d}x$ is an eigenfunctional of N_{op} and of N_ψ corresponding to the eigenvalue 1, as shown by (33) and the following equation:

$$N_\psi \int_{\Omega} \psi^* \Psi \, \mathrm{d}x = \tag{174}$$

$$= \int_{\Omega} \psi^*_{\mathrm{op}}(x) \Psi(x) \, \mathrm{d}x \int_{\Omega} \Psi^*(x') \psi_{\mathrm{op}}(x') \, \mathrm{d}x' \int_{\Omega} \psi^*(x'') \Psi(x'') \, \mathrm{d}x'' = \int_{\Omega} \psi^* \Psi \, \mathrm{d}x \,.$$

On the other hand, in the case of $\bar{\Delta}$, N_{op} and N_ψ do not have definite values, but their expectation values are $e/(e-1)$, with a normalized Ψ:

$$\frac{1}{e-1} (\bar{\Delta}[\psi^*, \Psi], N_{\mathrm{op}} \bar{\Delta}[\psi^*, \Psi]) = \frac{1}{e-1} \Big(\bar{\Delta}[\psi^*, \Psi], \Delta[\psi^*, \Psi] \int_{\Omega} \psi^* \Psi \, \mathrm{d}x \Big) = \tag{175}$$

$$= \frac{1}{e-1} \Delta[\Psi^*, \Psi] \int_{\Omega} \Psi^* \Psi \, \mathrm{d}x = \frac{e}{e-1}$$

$$\frac{1}{e-1} (\bar{\Delta}[\psi^*, \Psi], N_\psi \bar{\Delta}[\psi^*, \Psi]) = \tag{176}$$

$$= \frac{1}{e-1} \Big(\bar{\Delta}[\psi^*, \Psi], \int_{\Omega} \psi^*(x) \Psi(x) \, \mathrm{d}x \int_{\Omega} \Psi^*(x') \Psi(x') \, \mathrm{d}x' \, \Delta[\psi^*, \Psi] \Big) = \frac{e}{e-1} \,.$$

We have however,

$$(177)\qquad \frac{1}{e-1}\left(\bar{\Delta}[\psi^*, \Psi],\, LN_\Psi\, \bar{\Delta}[\psi^*, \Psi]\right) = \frac{1}{e-1}\left(\bar{\Delta}[\psi^*, \Psi],\, LN_{\psi_0}\, \bar{\Delta}[\psi^*, \Psi]\right) = 1\,,$$

Ψ being normalized.

The quantal superposition of states arises from an interesting property of the probability distribution of the linear manifolds in the Hilbert space. Let us consider the probability of the ray Ψ in the distribution corresponding to $\int_\Omega \psi^*\Psi\,\mathrm{d}x$:

$$(178)\qquad \Big(\int_\Omega \psi^*\Psi\,\mathrm{d}x,\, N_\Psi \int_\Omega \psi^*\Psi\,\mathrm{d}x\Big) = \Big(\int_\Omega \psi^*\Psi\,\mathrm{d}x,\, \int_\Omega \psi^*\Psi\,\mathrm{d}x\Big) = 1\,.$$

The probability of the ray corresponding to a function Ψ', which is not orthogonal to Ψ, in the same distribution is $\left|\int_\Omega \Psi^*\Psi'\,\mathrm{d}x\right|^2$

$$(179)\qquad \Big(\int_\Omega \psi^*\Psi\,\mathrm{d}x,\, N_{\Psi'} \int_\Omega \psi^*\Psi\,\mathrm{d}x\Big) =$$

$$= \Big(\int_\Omega \psi^*\Psi\,\mathrm{d}x,\, \int_\Omega \psi^*\Psi'\,\mathrm{d}x\Big) \int_\Omega \Psi'^*\Psi\,\mathrm{d}x = \left|\int_\Omega \Psi^*\Psi'\,\mathrm{d}x\right|^2,$$

since:

$$(180)\qquad N_{\Psi'} \int_\Omega \psi^*\Psi\,\mathrm{d}x = \int_\Omega \psi^*_{\psi_0}(x)\Psi'(x)\,\mathrm{d}x \int_\Omega \Psi'^*(x')\psi_{\psi_0}(x')\,\mathrm{d}x' \int_\Omega \psi^*(x'')\Psi(x'')\,\mathrm{d}x =$$

$$= \int_\Omega \Psi'^*(x')\Psi(x')\,\mathrm{d}x' \int_\Omega \psi^*(x)\Psi'(x)\,\mathrm{d}x\,.$$

7. – Discussion of the linear functionals χ.

We have seen in section 1 that the passage from the wave function $\Psi(x)$ to the corresponding linear functional $\int_\Omega \psi^*\Psi\,\mathrm{d}x$ and the inverse passage can be performed by means of the complex conjugated transformation functionals $\langle\psi^*|x\rangle$ and $\langle x|\psi\rangle$ respectively

$$(17)\qquad \chi[\psi^*] = \int_\Omega \langle\psi^*|x\rangle\Psi(x)\,\mathrm{d}x\,,$$

$$(21)\qquad \Psi(x) = \int \langle x|\psi\rangle\,\chi[\psi^*]\,\mathrm{d}\psi^*\,,$$

with $\langle \psi^* | x \rangle = \psi^*(x)$. The situation is different from that of the ordinary transformation theory, because we do not have operators for the quantities $(\psi(x))$ in the x-representation. The analogy with the ordinary transformation theory suggests that $\langle \psi^* | x' \rangle$ be the eigenfunctional of the operators of the quantities (x), corresponding to the eigenvalues x'. This is indeed true, since by taking $\Psi(x) = \delta(x - x')$ in (17) we get:

$$\langle \psi^* | x' \rangle = \int_\Omega \langle \psi^* | x \rangle \, \delta(x - x') \, \mathrm{d}x \,, \tag{181}$$

The above analogy suggests that $\langle x | \psi' \rangle$ be the eigenfunction of the operators for the quantities $(\psi(x'))$ in the x-representation, corresponding to the eigenvalues $\psi'(x')$. This requirement cannot be fulfilled because we do not have operators for the $(\psi(x'))$ in the x-representation.

We have seen in section 1 that the ordinary rule of interpretation of the transformation functions can be applied to the $\langle \psi^* | x \rangle$, $|\langle \psi^* | x \rangle|^2 \mathrm{d}x$ being the probability of finding values of the x in the ranges $x - x + \mathrm{d}x$, when the quantities $(\psi(x'))$ have the values $\psi(x')$, if it is assumed that the quantities $(\psi(x'))$ have the values $\psi(x')$ when Σ is in the state described by the normalized wave function ψ . The application of the above interpretation rule requires, of course, that the quantities $(\psi(x'))$ take only the values of normalized functions $\psi(x')$.

Let us consider now the wave function $\Phi(A')$ in a representation in which the complete set of commutable variables (A) is diagonal:

$$\Phi(A') = \int_\Omega \langle A' | x \rangle \Psi(x) \, \mathrm{d}x \,, \tag{182}$$

$$\Psi(x) = \int \langle x | A' \rangle \Phi(A') \, \mathrm{d}A' \,. \tag{183}$$

The integration with respect to the A' becomes a sum in the cases of discrete spectra of the A. Let us write:

$$\varphi(A') = \int_\Omega \langle A' | x \rangle \psi(x) \, \mathrm{d}x \,. \tag{184}$$

We have obviously

$$\chi[\psi^*] = \int_\Omega \Psi(x) \psi^*(x) \, \mathrm{d}x = \int_\Omega \Phi(A') \varphi^*(A') \, \mathrm{d}A' \,, \tag{185}$$

hence:

$$\langle \psi^* | A' \rangle = \varphi^*(A') \,. \tag{186}$$

Let us assume that the A have discrete spectra. The interpretation rule of the transformation theory shows that $|\langle\psi^*|A'\rangle|^2$ is the probability of the values $\psi^*(x)$ of the quantities $(\psi^*(x))$ when the A have the values A'. Since

$$\langle\psi^*|A'\rangle = \int_{\Omega} \psi^*(x)\langle x|A'\rangle\,\mathrm{d}x \qquad \left(\int |\langle x|A'\rangle|^2\,\mathrm{d}x = 1\right). \tag{187}$$

$\langle\psi^*|A'\rangle$ is the linear functional which corresponds to $\Psi(x) = \langle x|A'\rangle$. The above result shows that the interpretation rule of the transformation theory leads to the following interpretation rule for the linear functionals: $|\chi[\psi^*]|^2$ *is the probability of the state of* Σ *described by the normalized wave function* ψ^*, $\chi[\psi^*]$ *being normalized.* Thus we obtained from the transformation theory the probability distribution of the rays in the ψ Hilbert space discussed in section 6.

The theory of the linear functionals χ must be physically equivalent to the ordinary form of the quantum mechanics. There is however one thing which appers in a somewhat different form: the transition probabilities. In the ordinary formulation of the quantum mechanics there are no true transition probabilities from an initial state to the possible states at a later time. Thus the transition probability from the state ψ_0 at the time t_0 to the state $\psi_{A'}$ at the time t must be taken as the probability of obtaining the values A' in a measurement performed at the time t. There will be an effective transition only when the measurement is effectively performed, otherwise the system will be in the state $U(t)\psi_0$. In the χ-formalism there is a true transition probability, because probabilities are assigned to all the states at any instant of time, the transition probability from the state ψ_0 to the state $\psi_{A'}$ being the probability of $\psi_{A'}$ at the time t in the spontaneous evolution corresponding to $\chi[0;\psi^*] = \int_{\Omega} \psi^*\psi_0\,\mathrm{d}x$, i.e. to $|\chi[0;\psi_0^*]|^2 = 1$. Since $\chi[t;\psi_{A'}^*] = \int_{\Omega} \psi_{A'}^* U(t)\psi_0\,\mathrm{d}x$, the transition probability has the correct value: $\left|\int_{\Omega} \psi_{A'}^* U(t)\psi_0\,\mathrm{d}x\right|^2$.

There is no difficulty in extending the linear χ-formalism to the case of eigenfunctions $\psi_{A'}$ of the continuous spectrum of a hermitian operator A. We shall assume that the $\psi_{A'}(x)$ are normalized as follows:

$$\int_{\Omega} \psi_{A'}^*(x)\psi_{A''}(x)\,\mathrm{d}x = \delta(A' - A''). \tag{188}$$

$|\chi[\psi_{A'}]|^2\,\mathrm{d}A'$ gives the probability of the states ψ_α with α between A' and $A' + \mathrm{d}A'$.

We shall now consider the functional $\chi[\psi^*;\alpha]$

$$\chi[\psi^*,\alpha] = \int_{\Omega} \Psi_\alpha(x)\psi^*(x)\,\mathrm{d}x\,, \tag{189}$$

the functions $\Psi_\alpha(x)$ being normalized as the $\psi_{A'}$:

$$\int_\Omega \Psi^*_{\alpha'}(x)\Psi_{\alpha''}(x)\,\mathrm{d}x = \delta(\alpha' - \alpha'') \,. \tag{190}$$

Now $\chi[\psi^*; \alpha]$ [2] allows only to compute relative probabilities of different states ψ^*. An important case is that of $\Psi_\alpha(x) = \delta(x - \alpha)$. In this case $\chi[\psi^*, \alpha] = \psi^*(\alpha)$, so that the probability of a ψ is proportional to $|\psi(\alpha)|^2$.

8. – Second quantization interpretation of the functionals χ.

The second quantization of the Schrödinger equation (1) leads to the introduction of a quantized ψ-field whose quanta are systems Σ. Our χ-formalism is mathematically the same as the second quantization formalism for non interacting systems Σ treated as bosons, because of the sign minus in the commutation rules (8). The pure state of the system Σ described by the wave function $\Psi(x)$ corresponds to the state of the quantized ψ-field in which there is a single quantum Σ, the quantum being in the Σ-state Ψ. This state of the field is described by the linear functional $\int_\Omega \Psi(x)\psi^*(x)\,\mathrm{d}x$:

The linear functional (2) *obtained from the wave function* Ψ, *by going over to the dual of the* ψ^**-space, coincides with the wave functional of the state of the quantized* ψ*-field in which there is a single quantum* Σ, *the quantum being in the* Σ*-state* Ψ.

We shall consider now the state of the quantized ψ-field in which the $\psi_{\mathrm{op}}(x)$ have the eigenvalues $\Psi(x)$, i.e. the field state described by the wave functional $\Delta[\psi^*, \Psi]$. Since

$$\Delta[\psi^*, \Psi] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_\Omega \Psi(x_1) \dots \Psi(x_n)\psi^*(x_1) \dots \psi^*(x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \,, \tag{191}$$

we may write

$$\Delta[\psi^*, \Psi] = 1 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \left(\int_\Omega |\Psi(x)|^2\,\mathrm{d}x \right)^{n/2} \chi_n[\psi^*, \Psi] \,, \tag{192}$$

the $\chi_n[\psi^*, \Psi]$ being normalized eigenfunctionals of N_{op}:

$$\chi_n[\psi^*, \Psi] = \frac{1}{\sqrt{n!}} \left(\int_\Omega |\Psi|^2\,\mathrm{d}x \right)^{-n/2} \int_\Omega \Psi(x_1) \dots \Psi(x_n)\psi^*(x_1) \dots \psi^*(x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \,. \tag{193}$$

$\chi_n[\psi^*, \Psi]$ describes a n quantum state of the ψ-field, in which all the n quanta are in the Σ-state Ψ. It follows from (192) that the probability of finding n quanta in the field-state $\Delta[\psi^*, \Psi]$ is $(1/n!) \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] \left(\int_\Omega |\Psi|^2 \mathrm{d}x\right)^n$, i. e. the probabilities of the n are given by a Poisson law of average value $\int_\Omega |\Psi|^2 \mathrm{d}x$. This average value is simply the expectation value of N_{op} in the state $\Delta[\psi^*, \Psi]$:

$$(194) \qquad \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] (\Delta[\psi^*, \Psi], N_{\mathrm{op}} \Delta[\psi^*, \Psi]) =$$

$$= \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] \left(\Delta[\psi^*, \Psi], \int_\Omega \psi^*(x)\Psi(x)\,\mathrm{d}x\, \Delta[\psi^*, \Psi]\right) =$$

$$= \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] \left\{\int_\Omega \psi^* \Psi \,\mathrm{d}x\, \Delta[\psi^*, \Psi]\right\}_{(\psi=\Psi)} = \int_\Omega |\Psi|^2 \mathrm{d}x\,.$$

In the particular case of a normalized Ψ, the expectation value of N_{op} is simply 1. The expectation value of the number of quanta in the Σ-state described by the normalized wave function $\varphi(x)$ is $|\int_\Omega \Psi(x)\varphi^*(x)\,\mathrm{d}x|^2$:

$$(195) \qquad \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] (\Delta[\psi^*, \Psi], N_\varphi \Delta[\psi^*, \Psi]) =$$

$$= \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] \left(\Delta[\psi^*, \Psi], \int_\Omega \psi^*_{\mathrm{op}}(x)\varphi(x)\mathrm{d}x \int_\Omega \varphi^*(x')\psi_{\mathrm{op}}(x')\,\mathrm{d}x'\, \Delta[\psi^*, \Psi]\right) =$$

$$= \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] \int_\Omega \varphi^*(x')\Psi(x')\,\mathrm{d}x' \left(\Delta[\psi^*, \Psi], \int_\Omega \psi^*_{\mathrm{op}}(x)\varphi(x)\,\mathrm{d}x\, \Delta[\psi^*, \Psi]\right) =$$

$$= \left|\int_\Omega \Psi(x)\varphi^*(x)\,\mathrm{d}x\right|^2 .$$

When Ψ is normalized, the expectation value of N_φ coincides with the probability of the value 1 of p_φ in the Σ-state Ψ. It follows from (195) that the expectation value of $N_{\Psi_{\mathrm{nor}}}$ in the field-state $\Delta[\psi^*, \Psi]$ coincides with that of N_{op}:

$$(196) \qquad \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] (\Delta[\psi, \Psi], N_{\Psi_{\mathrm{nor}}} \Delta[\psi^*, \Psi]) = \int_\Omega |\Psi|^2 \mathrm{d}x\,,$$

$$(196a) \qquad \Psi_{\mathrm{ndr}}(x) = \Psi(x) \Big/ \left(\int_\Omega |\Psi|^2 \mathrm{d}x\right)^{1/2}.$$

Equation (99) shows that $\langle N'|\Psi\rangle$ is the wave function of the N' representation which corresponds to $\Delta[\psi^*, \Psi]$. It follows from (80) that the probability of finding N'_λ quanta Σ in the Σ-state φ_λ in the field state $\Delta[\psi^*, \Psi]$ is:

$$\exp\Big[-\int_\Omega |\Psi|^2 \mathrm{d}x\Big] \sum_{N'_\mu}{}^{(\mu\neq\lambda)} |\langle N'|\Psi\rangle|^2 = \tag{197}$$

$$= (N'_\lambda!)^{-1} \Big\{ \Big| \int_\Omega \Psi\varphi_\lambda^* \mathrm{d}x \Big|^2 \Big\}^{N'_\lambda} \exp\Big\{ -\Big| \int_\Omega \Psi\varphi_\lambda^* \mathrm{d}x \Big|^2 \Big\} .$$

Therefore the probability of finding n quanta Σ in the Σ-state φ is given by a Poisson distribution of average value $(\Delta[\psi^*, \Psi], N_\varphi \Delta[\psi^*, \Psi]) \exp\Big[-\int_\Omega |\Psi|^2 \mathrm{d}x\Big]$:

$$(n!)^{-1} \Big\{ \Big| \int_\Omega \Psi\varphi^* \,\mathrm{d}x \Big|^2 \Big\}^n \exp\Big[-\Big| \int_\Omega \Psi\varphi^* \,\mathrm{d}x \Big|^2 \Big] = \tag{198}$$

$$= (n!)^{-1} \Big\{ (\Delta[\psi^*, \Psi], N_\varphi \Delta[\psi^*, \Psi]) \exp\Big[-\int_\Omega |\Psi|^2 \,\mathrm{d}x\Big] \Big\}^n \cdot$$

$$\cdot \exp\Big\{ -(\Delta[\psi^*, \Psi], N_\varphi \Delta[\psi^*, \Psi]) \exp\Big[-\int_\Omega |\Psi|^2 \,\mathrm{d}x\Big] \Big\} .$$

The functional $\bar{\Delta}[\psi^*, \Psi]$ defined by (160) describes a state of the quantized ψ-field formed by the superposition of the states $\chi_n[\psi^*, \Psi]$, without a vacuum part:

$$\bar{\Delta}[\psi^*, \Psi] = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \Big(\int_\Omega |\Psi|^2 \,\mathrm{d}x \Big)^{n/2} \chi_n[\psi^*, \Psi] . \tag{199}$$

In the state $\bar{\Delta}[\psi^*, \Psi]$ the probability of finding n quanta Σ in the field is given by a modified Poisson law $(n!)^{-1} \Big(\int |\Psi|^2 \,\mathrm{d}x \Big)^n \Big\{ \exp\Big[-\int_\Omega |\Psi|^2 \,\mathrm{d}x) \Big] - 1 \Big\}^{-1}$.

Equation (195) shows that the expectation value of the number of quanta Σ in the Σ-state φ, when the ψ-field is in the state $\Delta[\psi^*, \Psi]$ with a normalized Ψ, coincides with the probability of obtaining the value 1 of the operator p_φ in a measurement performed on a system Σ in the Σ-state Ψ. More generally, the expectation value of any operator $\mathcal{A}$ in the field state $\Delta[\psi^*, \Psi]$ coincides with the expectation value of the quantity (A) in Σ-state Ψ, provided Ψ is normalized, as shown by (58) and (51). The above results show that the average behaviour of the ψ-field in the state $\Delta[\psi^*, \Psi]$, with a normalized Ψ, is the same as that of a system Σ in the state Ψ. When Ψ is not normalized, the average behaviour of the field is the same as that of an assembly of $\int |\Psi|^2 \mathrm{d}x$ systems Σ all in the state Ψ.

The wave functionals which satisfy (3) form a ring, more precisely a vector ring, because both the sum and the product of two solutions of (3) are solutions of (3). More generally let $f(u)$ be a holomorphic function of the complex variable u, we have:

$$\mathcal{H} f(\chi[t;\psi^*]) = \int_\Omega \mathrm{d}x\, \psi^*(x) H \frac{\delta}{\delta\psi^*(x)} f(\chi[t;\psi^*]) = \tag{200}$$

$$= f'(\chi[t;\psi^*]) \int_\Omega \mathrm{d}x\, \psi^*(x) H \frac{\delta}{\delta\psi^*(x)} \chi[t;\psi^*] = f'(\chi[t;\psi^*])\, \mathcal{H}\chi[t;\psi^*].$$

f' denoting the derivative of f. When $\chi[t;\psi^*]$ is a solution of (3) we have:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} f(\chi[t;\psi^*]) = \mathcal{H} f(\chi[t;\psi^*]). \tag{201}$$

It follows from this theorem that $\Delta[\psi^*, \Psi(t;x)]$ is a solution of (3). It is easily seen that any holomorphic function of s solutions $\chi^{(1)}[t;\psi^*],\ldots,$ $\chi^{(s)}[t;\psi^*]$ of (3) is also a solution of (3). Since

$$\Delta[\psi^*, \Psi^{(1)}]\, \Delta[\psi^*, \Psi^{(2)}] = \Delta[\psi^*, \Psi^{(1)} + \Psi^{(2)}], \tag{202}$$

the superposition of the pure states of Ψ gives rise to the following law of composition of the $\Delta[\psi^*, \Psi]$ field states:

$$\Delta\Big[\psi^*, \sum_i a_i \Psi^{(i)}(t;x)\Big] = \prod_i \big\{ \Delta[\psi^*, \Psi^{(i)}(t;x)] \big\}^{a_i}. \tag{203}$$

We have:

$$(\Delta[\psi^*, \Psi], \Delta[\psi^*, \Psi]) = \Delta[\Psi^*, \Psi] = \exp\Big[\int_\Omega |\Psi|^2 \mathrm{d}x\Big]. \tag{204}$$

Let us denote by P_Ψ the projection operator on the direction of the vector of the χ-space which corresponds to the field state $\Delta[\psi^*, \Psi]$:

$$\begin{cases} P_\Psi \chi[\psi^*] = \Delta[\psi^*, \Psi]\, (\Delta[\psi^*, \Psi], \chi[\psi^*]) \exp\Big[-\int_\Omega |\Psi|^2 \mathrm{d}x\Big], \\ P_\Psi \chi[\psi^*] = \chi[\Psi^*] \exp\Big[-\int_\Omega |\Psi|^2 \mathrm{d}x\Big]\, \Delta[\psi^*, \Psi]. \end{cases} \tag{205}$$

Since

$$\psi_{\mathrm{op}}(x) \chi[\psi^*] = \int \Psi(x) \Delta[\psi^*, \Psi]\, \chi[\Psi^*]\, \mathrm{d}\Psi^*, \tag{206}$$

we have:

$$\psi_{op}(x)\chi[\psi^*] = \int \exp\left[\int \Psi\Psi^* \, dx\right] \Psi(x) P_\Psi \chi[\psi^*] \, d\Psi^* . \tag{207}$$

We shall assume that the expectation value of $\psi_{op}(x)$ is $(\chi[\psi^*], \psi_{op}(x)\chi[\psi^*])$.

Since

$$(\chi[\psi^*], \psi_{op}(x)\chi[\psi^*]) = \int \Psi(x)(\chi[\psi^*], P_\Psi \chi[\psi^*]) \exp\left[\int_\Omega \Psi\Psi^* \, dx\right] d\Psi^* , \tag{208}$$

the probability of finding the values $\Psi(x)$ of the $\psi_{op}(x)$ may be taken tentatively as:

$$\mathcal{P}[\Psi] = (\chi[\psi^*], P_\Psi \chi[\psi^*]) \exp\left[\int_\Omega |\Psi|^2 \, dx\right] . \tag{209}$$

By taking into account (205), we get:

$$\mathcal{P}[\Psi] = |\chi[\Psi^*]|^2 . \tag{210}$$

This result is not satisfactory because the probabilities $\mathcal{P}[\Psi]$ may be larger than 1, as happens already in the case of linear functionals. In the case of linear functionals we can get satisfactory results from (210) by taking only normalized functions Ψ, as shown in section **7**.

The above result seems to indicate that, in the state of a field described by a non linear functional χ, it is not possible to get a satisfactory probability distribution for the values $\Psi(x)$ of the $\psi_{op}(x)$. This is not important because the $\psi_{op}(x)$ do not describe ordinary physical quantities of the systems Σ.

9. – Alternative interpretation of the χ-formalism for a single system Σ.

We have already given in section **6** a rule of physical interpretation for the χ-formalism considered as describing kinetic conditions of a single system Σ, by assigning probabilities to the linear manifolds M of the ψ Hilbert space. We shall now see that it is possible to modify the physical interpretation of the formalism of the quantized ψ-field in such a way that it will become a treatment of the motion of a single system Σ. We shall assume the following rules of physical interpretation:

(*a*) *The solutions of equation* (3), *such that* $\chi[t; 0] = 0$, *describe kinetic conditions of a system* Σ.

(*b*) *The possible values of a quantity* (A) *are the eigenvalues* A' *of the operator* A.

(*c*) *The physical quantity* (A) *is described by the corresponding field operator* $\mathscr{A}_{\text{nor}}$.

(*d*) *The expectation value of the quantity* (A) *of* Σ *in the kinetic condition described by the normalized functional* $\chi[\psi^*]$ *is given by the field expectation value* $(\chi[\psi^*], \mathscr{A}_{\text{nor}}\chi[\psi^*])$ *of* $\mathscr{A}_{\text{nor}}$.

(*e*) *The probability of finding the value* A' *of* (A) *in a measurement in the condition described by* $\chi[\psi^*]$ *coincides with the probability of obtaining the value* A' *of the quantity* $(A'p_{A'})$, $p_{A'}$ *being the projection operator of* A *for the eigenvalue* A'.

The field operator corresponding to $A'p_{A'}$ is $A'LN_{A'}$, in the present interpretation. Since the only eigenvalues of $A'p_{A'}$ are 0 and A', the expectation value $(\chi[\psi^*], A'LN_{A'}\chi[\psi^*])$ coincides with the probability of obtaining the value A' of $(A'p_{A'})$. This probability is the same as that of obtaining the value A' of (A), as a consequence of the rule (*e*). Thus we get the same result as in section 6.

The field operator LN_{op} corresponds to the quantity (1). Since

$$LN_{\text{op}}\chi[\psi^*] = \chi[\psi^*] \quad \text{when} \quad \chi[0] = 0\,, \tag{211}$$

the expectation value of LN_{op} is 1, as it should be. Therefore by considering only functionals such that $\chi[0] = 0$ and replacing the operator N_{op} for the total number of quanta by LN_{op} we get indeed a one quantum formalism.

We have:

$$\begin{aligned}(\bar{\Delta}[\psi^*, \Psi], LN_{A'}\bar{\Delta}[\psi^*, \Psi]) &= (\Delta[\psi^*, \Psi], LN_{A'}\Delta[\psi^*, \Psi]) = \\ &= \Big(\Delta[\psi^*, \Psi], L\Big\{\int_\Omega \psi^* p_{A'}\Psi\,\mathrm{d}x\,\Delta[\psi^*, \Psi]\Big\}\Big) = \\ &= \Big(L\Big\{\int_\Omega \psi^* p_{A'}\Psi\,\mathrm{d}x\,\Delta[\psi^*, \Psi]\Big\}\Big)_{(\psi^*=\Psi)} = \frac{\int_\Omega \Psi^* p_{A'}\Psi\,\mathrm{d}x}{\int_\Omega |\Psi|^2\,\mathrm{d}x}\Big\{\exp\Big[\int_\Omega |\Psi|^2\,\mathrm{d}x\Big] - 1\Big\}.\end{aligned} \tag{212}$$

Since

$$(\bar{\Delta}[\psi^*, \Psi], \bar{\Delta}[\psi^*, \Psi]) = \exp\Big[\int_\Omega |\Psi|^2\,\mathrm{d}x\Big] - 1\,, \tag{213}$$

the expectation value of $LN_{A'}$ in the condition described by $\bar{\Delta}[\psi^*, \Psi]$ coincides with the expectation value of $p_{A'}$ in the Σ-state Ψ, i.e. with the probability of finding the value A' in a measurement of (A) in the state Ψ. Thus in

the χ-formalism we can describe the pure state Ψ of Σ by $\bar{\Delta}[\psi^*, \Psi]$. The correspondence between the pure states Ψ of Σ and the functionals $\bar{\Delta}$ is remarkable because an eigenfunction $\psi_{A'}(x)$ of A corresponds to an eigenfunctional $\bar{\Delta}[\psi^*, \psi_{A'}]$ of $\mathcal{H}_{\text{nor}}$, with the same eigenvalue A':

$$\mathcal{H}_{\text{nor}}\bar{\Delta}[\psi^*, \psi_{A'}] = A'\bar{\Delta}[\psi^*, \psi_{A'}]\,. \tag{169}$$

The pure state Ψ of Σ can also be represented by the linear functional $\int_{\Omega}\Psi(x)\psi^*(x)\,\mathrm{d}x$ or by any of the homogeneous functionals

$$\frac{1}{\sqrt{n!}}\int_{\Omega}\Psi(x_1)\ldots\Psi(x_n)\,\psi^*(x_1)\ldots\psi^*(x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\,.$$

These functionals are also eigenfunctionals of $\mathcal{H}_{\text{nor}}$ corresponding to the eigenvalue A', when $\Psi = \psi_{A'}$:

$$\mathcal{H}_{\text{nor}}\int_{\Omega}\psi_{A'}(x_1)\ldots\psi_{A'}(x_n)\psi^*(x_1)\ldots\psi^*(x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n = \tag{214}$$

$$= A'\int_{\Omega}\psi_{A'}(x_1)\ldots\psi_{A'}(x_n)\psi^*(x_1)\ldots\psi^*(x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\,.$$

RIASSUNTO (*)

Si dimostra che l'ordinaria equazione di Schrödinger per un sistema dinamico Σ può essere sostituita da un'equazione più generale avente la forma dell'equazione di Schrödinger per un campo di Bose quantizzato i cui quanti sono i sistemi Σ. I funzionali d'onda lineari del campo quantizzato descrivono stati puri del sistema Σ e i funzionali d'onda non lineari descrivono, in generale, stati misti di Σ. La rappresentazione in cui gli operatori di emissione del campo di Σ quantizzato sono diagonali ha una parte preponderante nel presente formalismo. Si dimostra che gli autofunzionali degli operatori di assorbimento del campo di Σ quantizzato possono essere usati per dare una nuova descrizione degli stati di un sistema Σ, in quanto i valori di aspettazione delle grandezze del campo in autostati opportunamente scelti degli operatori di emissione coincidono coi valori di aspettazione delle corrispondenti grandezze negli stati puri di Σ, ma con maggiori fluttuazioni nel primo caso. Gli autofunzionali degli operatori di assorbimento hanno la notevole proprietà di essere elementi di matrice dell'operatore unitario del formalismo del campo e appaiono di natura più fondamentale che non i funzionali lineari corrispondenti all'ordinaria descrizione degli stati per mezzo delle funzioni d'onda di Σ.

(*) *Traduzione a cura della Redazione.*

An Investigation on Jets. (*)

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Summary. — The authors present a statistical method for the determination of the energy of the primary of a nucleon-nucleon collision in which secondary particles are produced. Such a method, which represents an appreciable improvement with respect to the procedures used by other authors, makes use of all the information that can be obtained from the measurements of the angles of the shower particles of a jet. The statistical error on the energy of the primary is given by a very simple formula and turns out to be always very large and increases with the anisotropy of the emitted particles in the center of gravity frame of reference. The method, which can be applied by successive approximations, is used to discuss 43 jets fue to p (28), n (6) and α (9) primaries. Besides the energy per nucleon of the primary of each jet, some information on the angular distribution of the emitted particles in the centre of gravity system, is deduced. The first approximation is based on the assumption that all the emitted particles have, in the centre of gravity frame of reference, the same velocity as the two colliding nucleons before the collision. A critical discussion of the determination of the energy of the primary shows that the result of this approximation has not in general much significance. The solution of second approximation needs the knowledge of the spectrum of the emitted particles and depends rather strongly on such a spectrum: if the experimental spectrum is reasonably well represented by a formula of the same type as that of Fermi's thermodynamical theory, the determination of the energy will be fairly satisfactory only if it is at least 50 GeV. But if the experimental spectrum can be represented for energies of the emitted particles much larger than $m_\pi c^2$, by a formula of the same type as that of Heisenberg, the evaluation of the energy of the primary will depend very strongly on the behaviour of the adopted spectrum at low energy also for ultrarelativistic collisions.

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1. – Introduction.

According to the terminology introduced by DANIEL et al. [1] we will call a « jet » an event consisting of a certain number n_s of shower particles but having very few, and in some cases, no, gray or black tracks. Such an event is usually interpreted as due to the collision of primary particles with a hydrogen nucleus or, more frequently, with a nucleon in the periphery of a nucleus [2].

In fact the absence, or almost absence, on heavily ionizing tracks due to the evaporation of an excited nucleus, is in favour of an almost negligible contribution to the observed event by secondary collisions.

If such an interpretation is correct the investigation on jets can give very useful information about the process of production of mesons in very high energy nucleon-nucleon collisions. In particular one can try to solve the problem of the dependence of the number n_s of shower particles on the energy of the primary. In fact an evaluation of the energy of the primary of a given jet can be made by means of simple measurements of the angles formed by the tracks of shower particles with the direction of the primary, making use of the obvious assumption that in a nucleon-nucleon collision the angular distribution of the emitted particles must be symmetrical with respect to the equatorial plane in the center of gravity frame of reference [2].

Starting from this point of view, various authors have evaluated the energy of the primary by different methods [1-8]: some have used the very simple and rough method of the median angle, others have tried to correlate the angles containing the percentages f and $1-f$ of the total number of particles.

These methods, however, do not use all the information that one can get from the angle measurements on a jet and do not allow an evaluation of the statistical error on the deduced value of the energy.

An extremely simple method satisfying these two requirements, has been developed in this laboratory during the last winter, and is presented in this

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paper. It can be applied by successive approximations. The first approximation is based on the assumption that all the emitted particles have the same velocity as the two colliding nucleons in the centre of gravity frame of reference.

The second approximation takes into account the spectrum of the produced particles, and therefore it needs the introduction of some special assumption: it represents however a considerable improvement with respect to some previous attempts to introduce a convenient average velocity of the produced particles ([6]).

Section 2 contains the principle of the method, section 3 its application in the special cases of the theories of Heisenberg and Fermi, section 4 the application of it to 43 jets observed in G5 emulsions exposed, for 7 hours at about 25 km, during the International Expedition held in Sardinia in June 1952; finally section 5 is a discussion of our results.

This work had already reached a fairly advanced stage when we received preprints first of a paper of HOANG TCHANG-FONG ([9]) and sometime later of DILWORTH, GOLDSACK, HOANG, SCARSI ([10]) dealing with the same problems.

These authors approach the problem, only in first approximation, with the method of the maximum likelihood; their experimental results are similar to ours and will be discussed in section 5.

The symbols used in the following are:

$c\beta$ = velocity of the centre of gravity in the laboratory frame of reference (L.S.)

$c\beta^*$ = velocity of the emitted particles in the centre of gravity frame of reference (G.S.)

$z = 1 + \eta = \beta/\beta^*$

θ = angle of emission of a particle observed in the L.S.

θ^* = angle of emission of a particle in the G.S.

$\mu = \cos\theta^*$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}; \qquad \beta(\gamma) = \sqrt{\frac{\gamma^2-1}{\gamma^2}}; \qquad \varepsilon^* = \frac{1}{\sqrt{1-\beta^{*2}}}$$

E_p = total energy of the primary in units Mc^2 in the L.S.

$E_p = 2\gamma^2 - 1.$

([9]) T. F. HOANG: *Journ. de Phys.*, **14**, 395 (1953).

([10]) C. DILWORTH, S. GOLDSACK, T. F. HOANG and L. SCARSI: *Compt. Rend.*, **236**, 1551 (1953).

2. – Principle of the method.

The problem is that of determining, for a given jet, the corresponding value of γ, making use of angle measurements and of the three following assumptions:

a) the considered jet is produced in a nucleon-nucleon collision;

b) the angular distribution of all the n_s particles emitted in the jet is symmetric with respect to the equatorial plane in the G.S.

c) the n_s shower particles are independent in the sense that there is no correlation between their angles nor between their energies of emission.

We note that assumption *b*) is a necessary consequence of assumption *a*) and that assumption *c*) is justified in spite of the theorems of conservation, because n_s is large and a considerable number of neutral particles is emitted together with the observed ones.

As usually done by various authors we start from the relation expressing the transformation of the angles from the G.S. to the L.S., namely:

$$\gamma = \frac{1}{|\operatorname{tg}\theta|}\,\frac{(1-\mu^2)^{1/2}}{|z+\mu|}\,. \tag{1}$$

Then we take the logarithm of both members

$$\ln\gamma = -x + u\left(\mu, \frac{\beta}{\beta^*}\right), \tag{2}$$

$$x = \ln|\operatorname{tg}\theta|\,, \tag{3}$$

$$u = \ln\frac{(1-\mu^2)^{1/2}}{|z+\mu|}\,, \tag{4}$$

we apply eq. (2) to the n_s tracks of the jet

$$\ln\gamma = -x_i + u(\mu_i, \beta_i^*)\,, \tag{5}$$

and we sum over all tracks: we get

$$\ln\gamma = -\frac{1}{n_s}\sum_1^{n_s} x_i + \frac{1}{n_s}\sum_1^{n_s} u(\mu_i, \beta_1^*)\,, \tag{6}$$

which is the basic relation of our method.

We note that the quantities x_i are the result of our measurements, while the quantities $u(\mu_i, \beta_i^*)$ are unknown and depend on the energy and angular distribution (in the G.S.) of the emitted particles.

2·1. – *First approximation.* — In order to make clear how eq. (5) allows us to determine γ let us treat it by a method of successive approximations: the first approximation, which we will call « spectrum independent », is obtained by introducing in (4)

$$\eta = \beta/\beta^* - 1 = 0 \,. \tag{7}$$

The variable

$$u_0 = \ln \frac{(1-\mu_0^2)^{1/2}}{1+\mu_0} = \ln \operatorname{tg} \frac{\theta^*}{2} \,, \tag{8}$$

satisfies the relations

$$\cos\theta^* = \mu_0 = \operatorname{tgh} u_0 , \qquad \sin\theta^* = (1-\mu_0^2)^{1/2} = 1/\cosh u_0 \,, \tag{9}$$

and can be treated, according to assumption *c*), as a casual variable; that means that we can ask what is the probability $F(u_0)\,\mathrm{d}u_0$ that its value falls in a given interval $\mathrm{d}u_0$. If we know the probability $f(\mu_0)\,\mathrm{d}\mu_0$ that one of the n_s tracks falls in a given solid angle $\mathrm{d}\mu_0$ (in the G.S.), one has obviously

$$F(u_0)\,\mathrm{d}u_0 = f(\mu_0)\frac{\mathrm{d}\mu_0}{\mathrm{d}u_0}\,\mathrm{d}u_0 = f(\operatorname{tgh} u_0)\,\frac{1}{\cosh^2 u_0}\,\mathrm{d}u_0 \,. \tag{10}$$

According to assumption *b*) $F(u_0)$ is an even function of u_0 nd therefore

$$\bar{u}_0 = \int_{-\infty}^{+\infty} u_0 F(u_0)\,\mathrm{d}u_0 = 0 \,. \tag{11}$$

Such a result is quite general and allows the determination of γ in the spectrum independent approximation because for $\eta = 0$ and n_s large the expectation value of the second term at the right hand side of eq. (6) is zero:

$$\frac{1}{n_s}\sum_1^{n_s} u_0(\mu_i) = 0 \,. \tag{12}$$

In other words we can rewrite eq. (5) in the form

$$\ln\gamma + x_i = u_{0i} \,, \tag{13}$$

and interpret it by saying that the variables n_{0i} fluctuate around their average value (11) with a variance given by

$$\sigma^2 = \overline{u_0^2} = \int_{-\infty}^{+\infty} u^2 F(u_0)\,\mathrm{d}u_0 \,. \tag{14}$$

In conclusion by making use of assumption *c*) and of eq. (12) (13) and (6)

we can write

$$\ln \gamma = -\frac{1}{n_s} \sum_{1}^{n_s} x_i \pm \frac{\sigma}{\sqrt{n_s}} , \tag{15}$$

which gives γ in first approximation. In deriving eq. (15) we make use of the Laplace theorem [11] which states that for n going to infinity the product of n functions having all the maximum 1 in the same point, tends to a gaussian function (see Appendix A). In order to be able to calculate σ by means of eq. (14) we need to know the angular distribution $f(\mu_0)$. Considering that any distribution satisfying assumption *b*) can always be represented by means of an even polynomial in μ_0, we have calculated (Appendix A, table A.1) the second moment

$$u_{0k}^2 = \sigma_k^2 = \xi_k^2 \sigma_0^2 , \tag{16}$$

corresponding to the normalized angular distribution function

$$f({}_k\mu_0) = \frac{2k+1}{2} \mu_0^{2k} , \tag{17}$$

for various values of k. If we know the experimental angular distribution and we express it by means of a convenient even polynomial in μ_0, we can write the standard deviation σ in the form

$$\sigma = \xi \sigma_0 , \qquad \sigma_0 = \frac{\pi}{\sqrt{12}} = 0.906 , \tag{18}$$

where ξ is a number of the order of 1 obtained in an obvious way as a combination of the ξ_k given in table A.1 of Appendix A.

Finally in order to determine in the first approximation the angular distribution in the G.S., we introduce in eq. (13) the value of γ obtained from eq. (15), and we calculate θ_i^* by means of eq. (8).

Our method has the disadvantage that if one track is observed at an angle $\theta_i = 0$, the corresponding x_i is $-\infty$ and γ diverges. It must be noted, however, that the probability that a particle is emitted exactly at $\mu_0 = +1$ is zero and that the used instrument does not permit the measurement of any angle θ_i less than a certain angle $\Delta\theta$. Therefore one cannot state that for a certain track the experimental result is $\theta_i = 0$ but only that for that track

$$\theta_i \leqslant \Delta\theta ; \tag{19}$$

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[11] See for instance, A. DUSCHEK: *Vorlesungen über die Höhere Mathematik*, Vol. II, pag. 235.

for such a track one has to use the following average value of x_i

$$\bar{x}_i = \frac{\int_0^{\Delta\theta} x\psi(\theta)\sin\theta\,\mathrm{d}\theta}{\int_0^{\Delta\theta}\psi(\theta)\sin\theta\,\mathrm{d}\theta}\,, \tag{20}$$

where $\psi(\theta)$ is the angular distribution of the tracks in the L.S. In all practical cases $\Delta\theta$ is so small that for $\theta \leqslant \Delta\theta$ one can assume $\psi(\theta) \sim$ constant. In this approximation eq. (20) reduces to

$$\bar{x}_i = \int_0^{\Delta\theta} \ln \operatorname{tg}\theta \sin\theta\,\mathrm{d}\theta \Big/ \int_0^{\Delta\theta} \sin\theta\,\mathrm{d}\theta \sim \ln \operatorname{tg}\Delta\theta - \frac{1}{2}\,.$$

Similar considerations can be made also for the second approximation.

2·2. – *Second approximation.* — The second approximation solution of eq. (6) is obtained by dropping the condition (7). That means that now we need to introduce some assumption about the form

$$S_\gamma(\varepsilon^*, \mu)\,\mathrm{d}\varepsilon^*\,\mathrm{d}\mu\,, \tag{22}$$

of the (normalized) spectrum of the particles emitted in the direction between μ and $\mu + \mathrm{d}\mu$; the index γ has been subscribed in order to emphasize that in general it will depend on γ.

As we will see in a moment we need to know the form of the spectrum (22) in order to calculate

$$\bar{\bar{u}} = \int_0^\infty \mathrm{d}\varepsilon^* \int_{-1}^{+1} S_\gamma(\varepsilon^*, \mu) u\,\mathrm{d}\mu\,, \tag{23}$$

$$\sigma'^2 = \bar{\bar{u}^2} = \int_0^\infty \mathrm{d}\varepsilon^* \int_{-1}^{+1} S(\varepsilon^*, \mu) u^2\,\mathrm{d}\mu\,. \tag{24}$$

It should be noted that the variables u and μ in the integrals (23), (24) are no longer connected by the simple eq. (8), but by the general eq. (4), in which the energies γ and ε^* appear in the variable z.

Going back to eq. (6) we note that it holds exactly for the actual values of θ_i, μ_i and β_i^* pertinent to each track of the considered jet. Therefore, for n_s large, the expectation value of the right-hand side of eq. (6) will be given by eq. (23). Therefore following an argument very similar to that given in

section 2·1, one gets, for the solution of second approximation,

$$\ln \gamma = -\frac{1}{n_s}\sum_1^{n_s} x_i + \bar{\bar{u}} \pm \frac{\sigma'}{\sqrt{n_s}} . \tag{25}$$

We note that this solution is exact under the assumptions *a*), *b*) and *c*). By comparing eq. (15) and (25) one gets

$$(\ln \gamma)_2 - (\ln \gamma)_1 = \bar{\bar{u}} , \tag{26}$$

which expresses the fact that the difference between the exact solution of our problem indicated by $(\ln \gamma)_2$ and the solution in the spectrum independent approximation (indicated in eq. (26) by $(\ln \gamma)_1$) is equal to the important quantity (23).

By subtracting eq. (13) from eq. (5) and using (26) one obtains another useful relation

$$u(\mu_i, \beta_i^*) = \bar{\bar{u}} + u_0(\mu_{0i}) . \tag{27}$$

Eq. (27) has the following meaning: once we have fixed the value of the angle of emission in the spectrum independent approximation (indicated as μ_{0i}), eq. (27) establishes the relation between the true value of the angle of emission μ_i and of the velocity $c\,\beta_i^*$.

Eq. (27) can also be written, making use of eqs. (4) and (8),

$$v(z_i, \mu_i) = e^{\bar{\bar{u}}} v(1, \mu_{0i}) , \tag{28}$$

where

$$v(z_i, \mu_i) = \frac{(1-\mu^2)^{1/2}}{z + \mu} . \tag{29}$$

3. – The second approximation in a few important cases.

In order to evaluate the error affecting the result of the spectrum independent approximation, we have thought it convenient to calculate the second approximation solution in a few important cases such as that of the theories of HEISENBERG [12] and FERMI [13].

According to the procedure given in the preceding section for such a cal-

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[12] W. HEISENBERG: *Zeits. f. Phys.*, **126**, 569 (1949).

[13] E. FERMI: *Progr. Theor. Phys.*, **5**, 570 (1950); *Phys. Rev.*, **81**, 683 (1951).

culation we need to know the spectrum of the particles emitted in a given direction (eq. (22)).

Considering, however, that these theories have the character of almost qualitative description of the phenomena involved, we have introduced a few rather rough approximations which simplify considerably the numerical calculations:

1) we assume that in eq. (22) the two variables ε^* and μ can be separated i.e. we put

$$S_\gamma(\varepsilon^*, \mu)\,\mathrm{d}\varepsilon^*\,\mathrm{d}\mu = S_\gamma(\varepsilon^*)\,\mathrm{d}\varepsilon^* f(\mu)\,\mathrm{d}\mu\,,$$

2) we assume that the angular distribution $f(\mu)$ is well represented by eq. (17) for a convenient value of k.

In order to get the expressions (23) and (24) we perform first the integration with respect to μ. Eq. (A6) of Appendix A gives the result of such a calculation. In fig. A.3 of Appendix A $\bar{u}_k(z)$ is plotted as a function of $\eta = z - 1$; if we put

$$\bar{u}_k(z) = A_k \cdot \eta\,, \tag{30}$$

with A_k = constant, we will not introduce too large an error and the integration with respect to the energy can be performed analytically; namely we have

$$\bar{u}_k = A_k F\,, \tag{31}$$

where

$$F = \beta(\gamma) \int_{\varepsilon\,\min}^{\varepsilon\,\max} \sqrt{\frac{\varepsilon^{*2}}{\varepsilon^{*2} - 1}}\, S_\gamma(\varepsilon^*)\,\mathrm{d}\varepsilon^* - 1. \tag{32}$$

3·1. – *The case of Heisenberg spectrum.* – The spectrum suggested by Heisenberg

$$S_\gamma(\varepsilon^*)\,\mathrm{d}\varepsilon^* = a(\gamma)\,\frac{\mathrm{d}\varepsilon^*}{\varepsilon^{*2}}\,, \qquad 1 \leqslant \varepsilon_0 \leqslant \varepsilon^* < \gamma\,, \tag{33}$$

$$a(\gamma) = \frac{\gamma\varepsilon_0}{\gamma - \varepsilon_0}\,, \tag{34}$$

is particularly uncertain at low energy and needs to be cut in some way. One can try to do that in different ways; for instance by introducing the volume

of the phase space by writing

$$S_\gamma(\varepsilon^*)\,\mathrm{d}\varepsilon^* = a(\gamma)\,\frac{\sqrt{\varepsilon^{*2}-1}}{\varepsilon^*}\,\frac{\mathrm{d}\varepsilon^*}{\varepsilon^{*2}}\,, \tag{35}$$

$$a^{-1}(\gamma) = \frac{1}{2}\left\{\arccos\frac{1}{\gamma} - \frac{\beta(\gamma)}{\gamma}\right\}, \tag{36}$$

or simply cutting the spectrum (33) at some convenient value ε_0 between 1 and 2.

All these procedures are arbitrary because, as stated by Heisenberg, the spectrum in the low energy region depends on the special form of the interaction. Therefore we have carried on the numerical calculations for the spectrum (33) and for three different values of the cut-off energy

$$\varepsilon_0 = 1.1 \qquad \varepsilon_0 = 1.5 \qquad \varepsilon_0 = 2.0$$

Introducing the spectrum (33) in eq. (32) one gets

$$F = \frac{\gamma\varepsilon_0}{\gamma-\varepsilon_0}\,\beta(\gamma)\left\{\arccos\frac{1}{\gamma} - \arccos\frac{1}{\varepsilon_0}\right\} - 1\,, \tag{37}$$

while from eq. (35) one gets

$$F = 2\beta(\gamma)\,\frac{\gamma-1}{\dfrac{\gamma}{\beta(\gamma)}\arccos\dfrac{1}{\gamma} - 1} - 1\,. \tag{38}$$

3·2. – *The case of Fermi spectrum.* — Unfortunately we have a simple expression for the energy spectrum only in the case of the thermodynamic approximation, which should be used only at very high energies. In spite of that we have extended the numerical calculations to the rather low energy region in which we are interested because we are now only trying to get an estimate of the error affecting the first approximation as it has been stated at the beginning of this section. In this case we write eq. (32) in the form

$$F = \beta(\gamma)\int_0^\infty \frac{\sqrt{\pi^2+1}}{\pi}\,S_\gamma(p)\,\mathrm{d}p - 1\,, \tag{39}$$

where

$$S_\gamma(p)\,\mathrm{d}p = \frac{1}{2{,}0404}\,\frac{x^2\mathrm{d}x}{e^x-1}\,; \qquad x = \frac{cp}{\tau}\,; \qquad p = m_\pi c\pi\,. \tag{40}$$

Assuming that in the collisions considered there is production only of pions and taking into account the conservation of angular momentum for a median collision ($\varrho = 0.959$), one gets

$$\frac{\tau}{Mc^2} = 0.122[\gamma(\gamma-1)]^{1/4}\,; \qquad x = 1.27\sqrt{\varepsilon^2-1}/[\gamma(\gamma-1)]^{1/4}\,,$$

which together with the development given in Appendix B allows us to calculate numerically $\bar{\bar{u}}_k$.

The numerical calculations have been performed for $k = 0$, i.e. for isotropic

TABLE I.

$\gamma =$	3	5	10	20	50	100	1 000
$\bar{\bar{u}}_F =$	.585	.479	.345	.237	.149	.106	.030
$\bar{\bar{u}}_{H_{1\cdot1}} =$	.783	.736	.691	.644	.625	.623	.621
$\bar{\bar{u}}_{H_{1\cdot5}} =$	.322	.345	.345	.322	.322	.322	.322

distribution (in the G.S.) of the emitted mesons. In Table I we give the values of $\bar{\bar{u}}$ for various γ and theories and in fig. 1 the quantity

$$-\frac{1}{n_s}\sum_1^{n_s} X_i\,; \qquad \text{with} \qquad X_i = \log_{10}|\operatorname{tg}\theta_i|\,,$$

is plotted as a function of γ as it results according to various approximations and theories: the curve marked S.I. corresponds to the first approximation « spectrum independent »: according to eq. (15) it is simply $\log\gamma$. The other three curves have been calculated using eq. (25) and represent an exact solution of the problem; the curve marked F corresponds to Fermi's theory, while the two curves marked H correspond to Heisenberg's theory and the two values 1.1 and 1.5 of the cut-off ε_0.

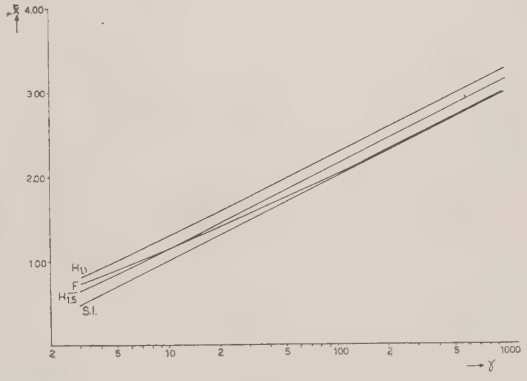

Fig. 1.

We have not plotted the curve corresponding to $\varepsilon_0 = 2$ because it differs very little from that for $\varepsilon_0 = 1.5$.

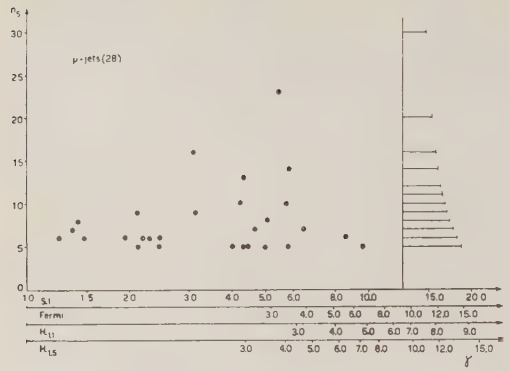

Fig. 2.

4. – Experimental results.

The events on which we report in this paper and which we will call jets in the following, have

$$n_s \geqslant 5 \qquad n_g + n_h \leqslant 4 \,.$$

Table II contains the data on 28 jets due to a single-charged primary (p-jets), 6 jets due a neutral primary (n-jets), and 9 due to α-particle (α-jets). In the case of the n-jets we have assumed that the direction of the primary is given by the axis of the shower particles defined as the direction of the resultant of unit vectors having the directions of the shower particles.

For each jet we give the value of γ deduced by the median angle and the value of γ deduced in the first approximation i.e. by means of eq. (15). Jets marked with a star have a track emitted at an angle larger than 90°. These tracks have obviously been neglected in the calculation of γ in the S.I. approximation but have to be considered in the calculations of second approximation. In fig. 2 we give n_s as a function of γ for the p-jets.

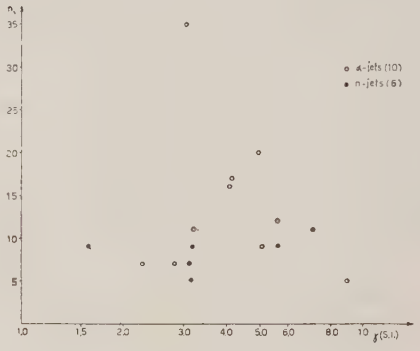

Fig. 3.

Besides the scale of γ corresponding to the spectrum independent approximation we give the scale corresponding to the second approximation; these have been deduced using the graphs of fig. 1. On the right side of fig. 2 we give the standard deviations due to the statistical fluctuations of the tracks as a function of n_s calculated according to eqs. (15) and (18) for isotropic angular distribution.

Fig. 3, similar to fig. 2, refers to the n- and α-jets.

In columns 7 to 16 of Table II we give the angular distribution in the G.S. deduced with the method of first approximation explained at the end of section 2·1; the data represent the number of tracks at minimum which fall in 10 given intervals of μ_0 ($\Delta\mu_0 = 0.2$). The same data are summarized in

TABLE II.

jet	n_s	n_g	n_h	$\gamma_{\theta_{1/2}}$	$\gamma_{\text{S.I.}}$	($\mu = +1$) Angular distribution ($\mu = -1$)									
						1	2	3	4	5	6	7	8	9	10
p-jets															
30	5	–	–	5.7	5.8 ± 2.4	–	–	1	–	–	2	2	–	–	–
48	5	–	–	5.1	5.0 ± 2.0	–	–	2	–	1	–	1	–	1	–
56	5	2	–	3.6	4.0 ± 1.6	–	–	1	1	–	1	2	–	–	–
25	5	1	–	5.7	4.4 ± 1.8	–	2	–	1	–	–	–	1	–	1
22	5	–	2	7.1	9.7 ± 4.0	–	1	–	–	1	–	1	2	–	–
64	5	–	2	3.1	2.1 ± 0.9	–	2	–	1	–	–	–	1	–	1
76	5	1	3	2.7	2.4 ± 1.0	2	–	–	–	1	–	–	–	1	1
36	6	1	3	1.2	1.5 ± 0.5	–	1	–	1	–	1	1	2	–	–
38	6	–	–	2.5	2.3 ± 0.8	–	–	2	1	–	1	1	–	1	–
*46	6	–	1	2.4	1.9 ± 0.7	–	1	–	1	1	1	–	–	–	1
*50	6	–	4	1.8	2.4 ± 0.9	–	2	–	–	–	–	1	1	1	–
54	6	–	1	4.5	8.6 ± 3.2	2	–	–	–	–	–	–	2	1	1
26	6	–	1	1.1	1.2 ± 0.5	2	–	–	–	–	1	–	2	–	1
66	6	–	3	2.0	2.2 ± 0.8	1	–	–	–	2	–	2	–	–	1
18	7	–	3	6.7	6.5 ± 2.2	1	1	1	–	1	–	1	–	–	2
20	7	–	–	3.5	3.4 ± 1.2	1	–	–	3	1	–	–	1	–	1
72	7	1	3	1.1	1.4 ± 0.5	1	–	–	1	–	–	2	3	–	–
80	7	4	–	3.5	4.6 ± 1.6	1	–	1	–	–	1	2	1	–	1
62	8	4	–	1.3	1.4 ± 0.4	1		2	1	–	2	–	–	1	1
70	8	2	3	4.8	5.0 ± 1.6	–	1	1	1	3	–	–	1	1	–
*29	9	–	3	1.4	2.1 ± 0.6	1	–	–	2	–	1	–	2	2	–
14	9	–	3	3.1	3.1 ± 0.9	–	1	1	–	2	1	–	1	3	–
40	10	–	2	6.3	5.8 ± 1.6	1	1	–	1	3	1	–	2	1	–
54bis	10	–	–	3.3	4.3 ± 1.2	3	–	–	–	1	–	1	1	2	2
13	13	–	3	3.2	4.3 ± 1.1	2	3	1	–	–	–	2	–	3	2
*17	14	–	3	4.9	5.9 ± 1.4	1	–	3	2	–	1	2	2	2	–
7	16	1	3	2.7	3.0 ± 0.7	2	–	3	1	1	2	2	4	–	1
60	23	–	4	6.0	5.5 ± 1.0	1	3	1	4	5	1	3	2	–	3
n-jets															
6	5	1	3	2.7	3.1 ± 1.3	–	–	1	–	1	1	2	–	–	–
8	7	–	1	3.7	3.1 ± 1.1	–	–	2	1	1	–	1	1	1	–
52	9	–	2	1.6	1.6 ± 0.5	–	2	1	1	1	1	2	–	–	1
82	9	–	4	4.0	5.6 ± 1.7	1	–	1	1	1	–	2	1	1	1
86	9	1	3	3.8	3.2 ± 0.9	–	1	1	2	3	–	–	1	–	1
5	11	–	–	7.1	7.1 ± 1.9	–	1	1	1	1	3	3	1	–	–
α-jets															
2	5	1	3	14.3	8.9 ± 3.6	1	1	1	–	–	–	1	–	–	1
28	7	2	3	2.4	2.8 ± 1.0	1	1	–	1	–	1	–	2	1	–
78	7	2	2	2.1	2.3 ± 0.8	1	1	–	–	1	1	1	1	–	1
58	9	1	1	6.3	5.1 ± 1.5	–	1	2	1	1	1	–	2	–	1
*12	11	2	–	3.0	3.2 ± 0.9	1	1	1	2	–	–	2	1	1	1
19	12	1	1	5.7	5.6 ± 1.5	1	2	1	2	–	1	1	1	3	–
10	17	1	2	4.0	4.1 ± 0.9	3	2	–	1	1	2	1	1	1	5
21	20	–	2	5.7	4.9 ± 1.0	2	3	4	–	2	–	–	2	5	2
3	35	–	3	4.9	4.8 ± 0.7	4	5	2	4	–	6	3	2	5	4

fig. 4, 5 and 6. These angular distributions are isotropic within the statistical errors as can be recognized using the Poisson law.

They have, however, only a conventional meaning.

If the velocities $c\ \beta_i^*$ of the single emitted particles were known, one could, of course, calculate, by means for instance of eq. (27) or (28) the corresponding true angles of emission μ_i. As that cannot be done at the moment, we can try to evaluate the influence of the spectrum of the angular distribution with respect to that of first approximation.

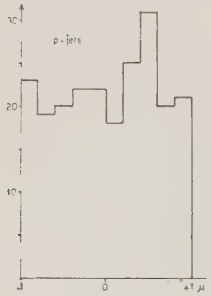

Fig. 4.

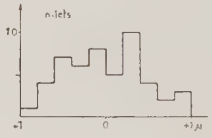

Fig. 5.

As an example we have calculated for $\bar{u} = -0.3$ (see Table I) the μ_i corresponding to 10 given values of μ_0 as a function of z (fig. 7).

For $z > 1$ for each value of θ_i there are 2 values of μ_i; in the graph of fig. 7 only the smaller one has been taken into consideration. Fig. 7 gives some

Fig. 6.

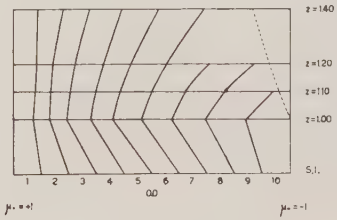

Fig. 7.

idea of the distortion of the angular distribution introduced by the use of the first approximation.

5. – Discussion.

The method developed in section 2·1 for the determination of γ in the first approximation has various obvious advantages over that used by other authors: it makes use of all the information that one can get from the simple measurement of the angles in a jet, it allows the evaluation of the standard deviation, in a correct way, and finally the operation of averaging the logarithm of the tangents of the observed angles does not seem to be more complicated than that involved for instance in the determination of the median angle.

The result however is affected by a rather large statistical error, which increases with increasing anisotropy as it is shown in table A.1 of Appendix A.

To such a statistical error it has to be added the experimental error due to the measurements of the angles. This is usually negligible with respect to the statistical error provided the absolute value of the measured angle is not very small. For small angles a special care has to be used in their determination.

From fig. 2 we see that our experimental points are distributed in a rather wide region above the line $n_s = 5$ which corresponds to the adopted definition of jet. Such a wide distribution can be due in part to variations of some parameter as for instance the impact parameter, in part to statistical fluctuations and finally in part to the contribution of secondary collisions whose effect could be different in different cases but is always in the sense to decrease γ and increase n_s.

Our results are confined to relatively low values of γ and therefore we cannot verify the existence of two well defined groups of jets as suggested recently by other authors ([8]) ([9]): jets whose multiplicity is low and increases very slowly with γ, and jets of rather high multiplicity which increases almost linearly with γ.

As shown in figs. 4, 5 and 6 in first approximation the angular distribution is isotropic at least in the energy region considered in the present paper. Although no quantitative statement can be made on the angular distribution of second approximation, the considerations presented at the end of section 4 seem to exclude a strong anisotropy.

The second approximation seems to be more satisfactory than some attempts made by other authors to consider deviation of z from 1. It has however the disadvantage of needing a definite assumption about the energy spectrum of the emitted particles and therefore loses the advantage of direct conclusion derived from the experimental data.

In spite of that it permits some remarks on the various possible theories.

From figs. 1 and 2 one can see that some doubt can be raised about the significance of the value of γ deduced in the spectrum independent approximation below and aroud $\gamma = 5$. In fact, also with a rather hard energy spectrum of pions, as that of Fermi's theory, the correction due to the velocity of the emitted pions is very large.

The situation is similar in the case of the Heisenberg theory, but it does not improve by going to very high energies. In this case for γ going to infinity the correction term $\bar{\bar{u}}$ tends to a constant but rather large value which depends on the adopted cut-off ε_0. In fact the variation of $\bar{\bar{u}}$ with γ is determined by two opposite effects: indeed an increase of γ introduces in the spectrum particles of higher energy and as a consequence the relative intensity

of the particles of low energy diminishes, but at the same time the less energetic mesons will be more strongly collimated in the forward direction.

In conclusion we think we have made clear what is the maximum information on jets that one can get only from measurements of angles. The weak point remains the basic assumption that jets are due to nucleon-nucleon collisions.

If only angle-measurements are available there is no possibility of checking such an assumption.

More informations on jets could be deduced by measurements of the momenta of the emitted particles. An effort in this direction is worth making only if it opens the possibility of checking the above mentioned assumption.

Our best thanks are due to Prof. E. AMALDI for stimulating discussions and valuable advice, and to Dr. G. BARONI for generous help. One of us (D.M.) is indebted to the Italian Government for a scholarship and has pleasure in expressing his acknowledgments.

APPENDIX A.

In order to make clear the meaning of eq. (15) let us first consider the case of isotropic angular distribution in the G.S. The corresponding distribution function of the variable u_0, $F(u_0)$, is given in fig. A.1 from which one can recognize that it has a behaviour very similar to that of a gaussian distribution. Therefore one can say that a single value x_i represents an independent measurement of $\ln \gamma$ with variance σ. Eq. (15) is justified by the fact that if the number of observations n_σ is large, for the Laplace theorem, the distribution of the corresponding values of $\ln \gamma$ will tend to a normal distribution.

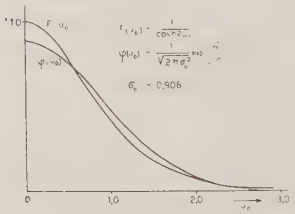

Fig. A.1.

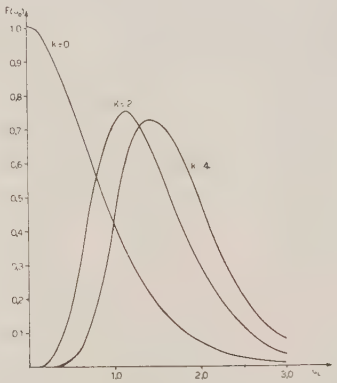

Fig. A.2.

In the case of non-isotropic angular distribution the distribution function $F(u_0)$ depends on the value of k. As examples we plotted in fig. A.2, $F(u_0)$ for $u_0 \geqslant 0$ and for $k = 0, 2, 4$. Of course $F(u_0)$, being an even function of u_0, is symmetrical with respect to the origin.

Also in this case one can apply the Laplace theorem but one could suspect that the normal distribution is obtained only for an extremely large number (at the limit an infinite number) of independent observations x_i. That is not true because, for each value of k, the distribution function has a behaviour, for $u_0 \geqslant 0$ (as well as for $u_0 \leqslant 0$) very similar to that of a gaussian function (fig. A.2). Therefore one can repeat the consideration made for the isotropic case, provided we take into account all possible repartitions of the n_s observations between the two « almost normal distributions » corresponding respectively to $u_0 \geqslant 0$ and $u_0 \leqslant 0$.

One can immediately recognize that the variance obtained by combining the variance corresponding to the « almost normal distribution » for $u_0 \geqslant 0$, with that due to the fluctuations of the numbers of observations m and $n_s - m$ falling in the two regions $u_0 \geqslant 0$ and $n_0 \leqslant 0$, is identical with the second moment of $F(u_0)$ for u_0 varying between $-\infty$ and $+\infty$.

We report here the derivation of a few integrals given to us by Dr. W. Gross of the Istituto Nazionale per le Applicazioni del Calcolo:

$$\overline{u_{0k}^2} = \frac{2k+1}{2} \int_{-1}^{+1} \mu^{2k} \ln^2 \frac{(1-\mu^2)^{1/2}}{1+\mu} \, \mathrm{d}\mu = \frac{1}{2} I_k , \tag{A.1}$$

$$I_k = (2k+1) \int_{-1}^{+1} x^{2k} \ln^2 \sqrt{\frac{1-x}{1+x}} \, \mathrm{d}x ,$$

$$I_k - I_{k-1} = \int_{-1}^{+1} (x^{2k+1} - x^{2k-1})' \ln^2 \sqrt{\frac{1-x}{1+x}} \, \mathrm{d}x = \int_{-1}^{+1} x^{2k-1} \ln \frac{1+x}{1-x} \, \mathrm{d}x =$$

$$= \frac{1}{2k} \int_{-1}^{+1} (x^{2k} - 1)' \ln \frac{1+x}{1-x} \, \mathrm{d}x = \frac{1}{k} \int_{-1}^{+1} \frac{x^{2k}-1}{x^2-1} \, \mathrm{d}x = \frac{1}{k} \sum_{0}^{k-1} \int_{-1}^{+1} x^{2i} \mathrm{d}x .$$

And therefore

$$I_k = I_{k-1} + \frac{2}{k} \sum_{0}^{k-1} \frac{1}{2i+1} , \tag{A.2}$$

$$I_0 = \int_{-1}^{+1} \ln^2 \sqrt{\frac{1-x}{1+x}} \, \mathrm{d}x = 2 \int_{0}^{1} (x-1)' \ln^2 \sqrt{\frac{1-x}{1+x}} \, \mathrm{d}x = 2 \int_{0}^{1} \frac{1}{1+x} \ln \frac{1+x}{1-x} \, \mathrm{d}x .$$

With $z = (1-x)/(1+x)$ one gets

$$I_0 = -2 \int_{0}^{1} \frac{\ln z}{z+1} \, \mathrm{d}z = 2 \frac{\pi^2}{12} . \tag{A.3}$$

From (A.1), (A.2) and (A.3) we get finally

$$\sigma_k^2 = \overline{u_{0k}^2} = \frac{\pi^2}{12} + \sum_1^k \frac{1}{i} \sum_0^{i-1} \frac{1}{2s+1} \,. \tag{A.4}$$

In table A.1 we give the values of σ_k^2 and ξ_k calculated from eq. (A.4) for a few values of k.

TABLE A.1.

k	σ_k^2	ξ_k
0	$\pi^2/12$	1.00
1	$\pi^2/12 + 1$	1.49
2	$\pi^2/12 + 5/3$	1.74
3	$\pi^2/12 + 98/45$	1.91
4	$\pi^2/12 + 2454/945$	2.04

For the calculation of $\bar{\bar{u}}$ we need the following integral

$$\bar{u}_k(z) = \frac{2k+1}{2} \int_{-1}^{+1} \mu^{2k} \ln \frac{\sqrt{1-\mu^2}}{z+\mu} \, \mathrm{d}\mu = \frac{2k+1}{2} J_k \,, \tag{A.5}$$

$$J_k = \int_{-1}^{+1} x^{2k} \ln \frac{\sqrt{1-x^2}}{z+x} \, \mathrm{d}x = J_{k_1} - J_{k_2} \,,$$

$$J_{k_1} = \int_{-1}^{+1} x^{2k} \ln \sqrt{1-x^2} \, \mathrm{d}x = \frac{2}{2k+1} \left[\ln 2 - \sum_0^k \frac{1}{2i+1} \right] ,$$

$$J_{k_2} = \int_{-1}^{+1} x^{2k} \ln |z+x| \, \mathrm{d}x = \frac{1}{2k+1} \left\{ \ln |z^2-1| - \int_{-1}^{+1} \frac{x^{2k+1}}{|z+x|} \, \mathrm{d}x \right\} ,$$

$$x^{2k+1} = -z^{2k+1} + (x+z) \sum_0^{2k} (-1)^i x^i z^{2k-i}$$

$$\int_{-1}^{+1} \frac{x^{2k+1}}{|x+z|} \, \mathrm{d}x = z^{2k+1} \ln \left| \frac{z+1}{z-1} \right| - 2 \sum_0^k \frac{z^{2(k-i)}}{2i+1} \,.$$

And therefore

$$\bar{u}_k(z) = \sum_0 \frac{z^{2(k-\cdot)}}{2i+1} - \sum_0^{k-1} \frac{1}{2i+1} - \frac{1}{2} z^{2k+1} \ln \left| \frac{z+1}{z-1} \right| - \frac{1}{2} \ln |z^2-1| + \ln 2 \,. \tag{A.6}$$

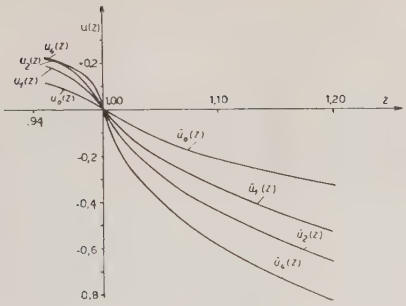

Fig. A.3.

In fig. A.3 we plotted u_k as a function of $\eta = z - 1$ for $k = 0, 1, 2, 4$. From these graphs we have obtained the values of A_k listed in table A.2.

TABLE A.2.

k	0	1	2	4
$-A_k$	2.4	3.8	5	10

APPENDIX B.

Let us write the integral of eq. (39) in the form

$$J_{(\gamma)} = \frac{1}{B}\int_0^\infty \sqrt{1+(Bx)^2}\,\frac{x\,\mathrm{d}x}{e^x-1}\,, \qquad B(\gamma) = \pi/x \tag{B.1}$$

and use the approximate expressions

$$\begin{cases} \sqrt{1+(Bx)^2} = 1 + \frac{1}{2}B^2x^2\,, & x < \frac{1}{B} \\ \sqrt{1+(Bx)^2} \doteq 1 + Bx\,, & x > \frac{1}{B} \end{cases} \tag{B.2}$$

Then one has

$$J_{(\gamma)} = \frac{1}{2B}\left[G_1(\infty) + G_1\left(\frac{1}{B}\right)\right] + G_2(\infty) - G_2\left(\frac{1}{B}\right) + \frac{B}{2}\,G_3\left(\frac{1}{B}\right), \tag{B.3}$$

where

$$G_k(z) = \int_0^z \frac{x^k}{e^x-1}\,\mathrm{d}x = \frac{1}{2}\int_0^z x^k \operatorname{ctgh}\frac{x}{2}\,\mathrm{d}x - \frac{1}{2}\,\frac{z^{k+1}}{k+1}\,, \tag{B.4}$$

which can be written, as kindly suggested to us by Prof. G. PLACZEK in the form

$$G_k(z) = \frac{z^k}{2}\left[\frac{2}{k} - \frac{z}{k+1} + \frac{z^2}{6(k+z)} - \frac{z^4}{360(k+4)} + \frac{z^6}{15\,126(k+6)}\right]. \tag{B.5}$$

RIASSUNTO

Viene proposto un metodo statistico per la determinazione dell'energia del primario di un urto nucleone-nucleone in cui vengono prodotte più particelle secondarie. Tale metodo, che costituisce un notevole perfezionamento rispetto ai procedimenti seguiti da altri autori, fa uso di tutte le informazioni che si possono trarre da misure angolari eseguite su di un getto e fornisce una semplice espressione per l'errore statistico da cui è affetto il risultato, la cui entità, sempre assai elevata, viene a dipendere dalla distribuzione angolare dei corpuscoli emessi nel sistema del baricentro. Il metodo, che si può applicare per approssimazioni successive, viene usato per la discussione di 43 getti dovuti a primari p (28), n (6) ed α (9). Oltre all'energia per nucleone del primario di ciascun getto, vengono dedotte alcune informazioni sulla distribuzione angolare dei corpuscoli emessi nel sistema del baricentro. Un esame critico della determinazione dell'energia del primario mostra che il risultato della prima approssimazione basata sull'ipotesi che tutti i corpuscoli emessi abbiano, nel sistema del baricentro, la stessa velocità che posseggono i due nucleoni prima dell'urto, non ha in generale molto significato. La soluzione di seconda approssimazione richiede invece la conoscenza dello spettro dei corpuscoli emessi e dipende abbastanza fortemente da questo. Se lo spettro sperimentale fosse ben rappresentato da una formula simile a quella della teoria termodinamica di Fermi, la determinazione dell'energia avrebbe un certo senso solo nel caso in cui essa fosse superiore ad almeno 50 GeV. Ma se lo spettro sperimentale fosse rappresentato, per energie dei mesoni emessi grandi rispetto all'energia di quiete, da una espressione del tipo di quella di Heisenberg, allora la valutazione dell'energia del primario verrebbe a dipendere fortemente dall'andamento dello spettro usato alle basse energie anche per urti ultrarelativistici.

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On the Scattering of μ-Mesons by Nuclei.

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(ricevuto il 9 Settembre 1953)

Summary. — Recent experiments have lead to unexpectedly large cross sections for anomalous scattering of μ-mesons. It is here examined whether a better evaluation of the electromagnetic contributions can diminish this discrepancy. The incoherent electric cross section and the dependence on the nuclear model employed are here considered in detail. To calculate the incoherent contribution the closure approximation is used: a discussion is given of its validity and in some cases the results are compared whith those of calculations which do not use the closure approximation. It is concluded that the incoherent contribution cannot explain the large anomalous cross sections.

The evidence for a weak interaction between μ-mesons and nucleons is based essentially on experiments on the capture of negative μ-mesons, underground experiments on cosmic rays, and scattering experiments. Experiments of the first type allow a theory of the interaction between μ-mesons and nucleons with a very small coupling constant to be established; experiments of the second type lead to values of the cross-sections which are explicable entirely on the basis of electromagnetic interaction; among the experiments of the third type, some have confirmed these results, while others, more recent, seem to indicate some deviations. The very accurate measurements of Amaldi and Fidecaro [1], performed with iron, have led to upper limits for the cross-sections of anomalous μ-meson nucleon scattering of the order of $4{,}5 \cdot 10^{-29}$ cm^2 per nucleon and $2{,}3 \cdot 10^{-30}$ cm^2 per nucleon for μ-mesons whose kinetic energy after collision is between 200 and 320 MeV or > 320 MeV respectively; and

(1) E. Amaldi and G. Fidecaro: *Nuovo Cimento*, **7**, 535 (1950).

results of the same order have recently been obtained by WALKER [2] with carbon. Experiments performed with heavier nuclei by GEORGE and TRENT [3] (1951 lead), WHITTEMORE and SHUTT [4] (1952, lead), GEORGE, REDDING and TRENT [5] (1953, lead), KANNANGARA and SHRIKANTIA [6] (1953, emulsion nuclei), LEONTIC and WOLFENDALE [7] (1953, lead) all seem to indicate larger cross-sections ($2 \cdot 10^{-28}$ cm^2 per nucleon) for anomalous scattering of μ-mesons on nucleons. It has, moreover, been explicitly observed that such a scattering could be interpreted as coherent Coulomb scattering alone, only on the hypothesis (which here aims to be only of an exemplifying nature) of a point nucleus. Since, on the other hand, the scheme of actual theoretical knowledge of the interaction between μ-mesons and nucleons entirely excludes the possibility of a non-electric interaction between μ-mesons and nucleons such to explain the cross-sections observed, and since there are, moreover, more general theoretical arguments which support this scheme, a more accurate evaluation is needed, first of all, of the various contributions to the scattering which are still of an electromagnetic origin or which depend sensibly on hypotheses on the structure of the nucleus or on hypotheses on the structure of the nucleon.

The electromagnetic cross-section can be considered as the sum of two contributions: a coherent contribution corresponding to the collisions without excitation of the nucleus struck, and an incoherent contribution corresponding to the collisions followed by excitation of the nucleus struck. For light nuclei, the calculation of the two contributions, performed by AMALDI, FIDECARO and MARIANI [8], has shown that the incoherent contribution is far from negligible, being in fact predominant at large angles. Moreover, such a contribution results in bringing the cross-section of a nucleus of finite dimensions nearer to the cross-section of the same nucleus regarded as puntiform.

In an independent-particle model the incoherent contribution arises, to a first approximation, from the independent transitions of the single nucleons from their original state to excited states. Although for light nuclei the number of different transitions taken into account is still such as to allow a direct evaluation of the cross-section, for heavy nuclei the number of different transitions to be considered is such as to necessitate the use of some approximation. The nature of the approximation used in the present work (closure

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[2] W. O. WALKER: (1953) not published.

[3] E. P. GEORGE and P. T. TRENT: *Proc. Phys. Soc.*, A **64**, 1134 (1951).

[4] W. L. WHITTMORE ans R. P. SHUTT: *Phys. Rev.*, **88**, 1312 (1952).

[5] E. P. GEORGE, J. L. REDDING and P. T. TRENT: *Proc. Phys. Soc.*, A **66**, 533 (1953).

[6] M. L. T. KANNANGARA and G. S. SHRIKANTIA: *Phil. Mag.*, **44** (1953).

[7] B. LEONTIC and A. WOLFENDALE: to be published (1953).

[8] E. AMALDI, G. FIDECARO and F. MARIANI: *Nuovo Cimento*, **7**, 553, 758 (1950).

approximation) consists in assuming that the preponderant contribution to the cross-section arises from transitions in which the loss of energy of the μ-meson is still small with respect to the initial energy possessed by the same incident μ-meson. It is therefore clear that the validity of this approximation will increase with the energy of the incident μ-meson. One can also maintain within the same limits that the contribution of those transitions which have strictly not to be considered because they do not conserve energy, is a contribution which, added to the contribution of those transitions which do conserve energy, causes only negligible modifications, which become more negligible as the initial energy of the μ-meson increases. Thus, summing over all possible transitions, whether they do or do not conserve energy, it is possible to carry out the calculation of the cross-section using hypotheses referring only to the nucleus in its ground-state, thus avoiding hypotheses on the excited states of the nucleus. This avoids the discussion of the further question of establishing between what limits and in what way different hypotheses on the excited states can alter the result for the incoherent cross-section. One could not avoid examining this point since it is well known that, although for the ground state definite indications, in part theoretical, exist, which allow the choice of the model to be decided, such indications are not so clear for the excited states.

Here and in the following Ψ_0 indicates the wavefunction of the nucleus in its ground state, $\boldsymbol{R}_i$ the position vector of the i-th proton, $k = 2K_0 \sin(\theta/2)$ the modulus of the difference between the vector $\boldsymbol{K}_0$ which represents the initial wave number of the incident μ-meson and the vector $\boldsymbol{K}$ which represents the final wave number of the same meson deflected by an angle θ, and it is assumed that $\boldsymbol{K}$ has still the same modulus as $\boldsymbol{K}_0$. The ratio between the scattering by a nucleus of atomic number Z, of finite dimensions, and the scattering by a hypothetical nucleus of the same Z, regarded as a point nucleus, can thus be expressed by means of the form factor:

$$(1) \qquad \mathcal{C} + \mathcal{J} = Z^{-2}\left[Z + 2\int \Phi_0^* \sum_{i>k}^{Z}\sum^{Z} \cos \boldsymbol{k}(\boldsymbol{R}_i - \boldsymbol{R}_k)\Phi_0 \, d\tau\right],$$

supposing to have integrated over the coordinates of the neutrons and thus substituted a certain Φ_0 for Ψ_0. This expression is of the form $1 - O(k^2)$ for small angles, while it falls rapidly to the value Z^{-1} for large angles owing to the sinusoidal terms contained in the integral, which oscillate rapidly for large values of k. The physical significance of this behaviour is expressed in the fact that detailed knowledge of the nucleus is unimportant at small angles and at large angles. For small angles the scattering can still be confused with the Rutherford scattering of a point nucleus (for angles less than the ratio between the wave lenght of the μ-meson and the radius of the atom the

screening effect thas to be taken into account), for large angles the nucleus may be regarded as broken up into its components and scatters in the same way as Z distinct scattering centres. In this latter zone the knowledge of the structure of the nucleon is now more important for the evaluation of the cross-section than the knowledge of the structure of the nucleus. In the intermediate angles, the cross-section depends essentially on the properties of the wave function Ψ_0, that is, on the details of the charge-distribution in the nucleus, and on the correlation properties of its particles.

If a representation in the momentum-space is chosen where the nucleus in its ground state is described by a certain χ which depends on all the momenta $\boldsymbol{k}_i$ of the single nucleons, the integral in (1) takes the form

$$(2) \quad \frac{1}{2}\sum_{i=1}^{Z}\sum_{k=1}^{Z}\iint\ldots\int \mathrm{d}\boldsymbol{K}_1\,\mathrm{d}\boldsymbol{K}_2\ldots\mathrm{d}\boldsymbol{K}_Z\chi^*(\boldsymbol{K}_1,\boldsymbol{K}_2,\ldots,\boldsymbol{K}_i+\boldsymbol{k},\ldots,\boldsymbol{K}_k-\boldsymbol{k},\ldots,\boldsymbol{K}_Z)\cdot \cdot\chi(\boldsymbol{K}_1,\boldsymbol{K}_2,\ldots,\boldsymbol{K}_Z).$$

This shows first of all that by using the closure approximation, one succeeds, from a formal point of view, in replacing a problem in which the interaction between the μ-meson and the single nucleons must be considered, by a problem in which the nucleons interact directly among themselves; secondly it shows how a large cross-section at large angles can be obtained only with models of the nucleus in which components of very large momenta are present.

In a Fermi model there are assumed to be present all momenta less than a well defined maximum momentum and no momenta greater than this maximum. Corresponding to this precise definition of a maximum momentum, a precise definition is obtained in this model of a minimum angle $\bar{\theta}$, starting from which the single protons scatter rigorously as Z distinct centres. This angle depends on the energy of the incident μ-meson, clearly decreasing with increase of energy, and does not depend upon A, but only very slightly on the ratio Z/A. The independence of $\bar{\theta}$ on A reflects the well known fact that in a nucleus, the maximum momentum which the single nucleons can have does not depend on A, this being a property which comes from the density saturation. It can be said intuitively that large values of k correspond to a large value of the recoil momentum given up by the meson, and only when this momentum, apart from factors of the order unity, is equal to or greater than the maximum momentum which nucleons can have in the nucleus, do the single nucleons begin to scatter as Z independent centres. Table I shows, for the purpose of illustration, the values of the angle $\bar{\theta}$ above which it can be maintained that the nucleons scatter in an independent manner.

TABLE I.

E_{MeV}	λ in cm	λ/r_0 $r_0 = 1.41 \cdot 10^{-13}$ cm	$\bar{\theta}$ for light nuclei $N = Z$	$\bar{\theta}$ for heavy nuclei $N = 1{,}5\ Z$
200	$1.17 \cdot 10^{-13}$	0.83	—	—
400	$0.51 \cdot 10^{-13}$	0.36	66° 54′	61° 34′
600	$0.39 \cdot 10^{-13}$	0.24	42° 52′	39° 38′
800	$0.25 \cdot 10^{-13}$	0.18	31° 10′	28° 50′
1000	$0.20 \cdot 10^{-13}$	0.14	24° 38′	22° 50′

It is clear that for nuclear models for which no such maximum momentum is defined, there will not be defined precisely a minimum angle above which the single nucleons scatter independently and in general the manner in which the cross-section for large angles tends to the limiting value Z^{-1} will reflect the manner in which the probability of the large momenta tends to zero in the nuclear model chosen. If moreover there exist components of infinitely large momenta the form factor (1) instead of approaching the limiting value Z^{-1} for large angles, will approach a limiting value different from Z^{-1} and the greater the components of these infinitely large momenta the further will be such value from Z^{-1}.

The form factor (1) contains the coherent contribution $\mathcal{C}$ and the incoehrent contribution $\mathcal{I}$ not yet separated. If the hypothesis is made of an independent particle model for the nucleus in its fundamental state, a criterion for again separating the coherent and incoherent contributions in (1) can be obtained according to the observations made concerning (2), where it was noted that the closure approximation led in a certain sense to replace the problem of the interaction between μ-mesons and single nucleons by a problem in which the single nucleons interact among themselves. And so the following criterion is immediately formulated: the direct terms relative to such an interaction for the nucleons among themselves constitute the coherent scattering, while the exchange terms contribute to the incoherent scattering. The application of this criterion gives a coherent scattering which depends only on the charge distribution and whose expression is rigorous in the limits of Born's approximation and of the nuclear model employed; while possible errors connected with the use of the closure approximation remain solely in the terms relative to the incoherent scattering.

In this way it can be easily demonstrated that the form factor relative to the incoherent scattering for all angles certainly assumes values less than the limiting value Z^{-1}. This result is valid for a generic independent particle model and certainly holds for a shell model; but it seems improbable that it

would need to be appreciable modified for models of a different nature, and in this connection, the fact that such a model is assumed only with regard to the ground state, leaving the nature of the excited states unspecified, gives one greater confidence in the result. This result therefore allows it to be excluded that the large cross-sections of anomalous scattering in heavy nuclei

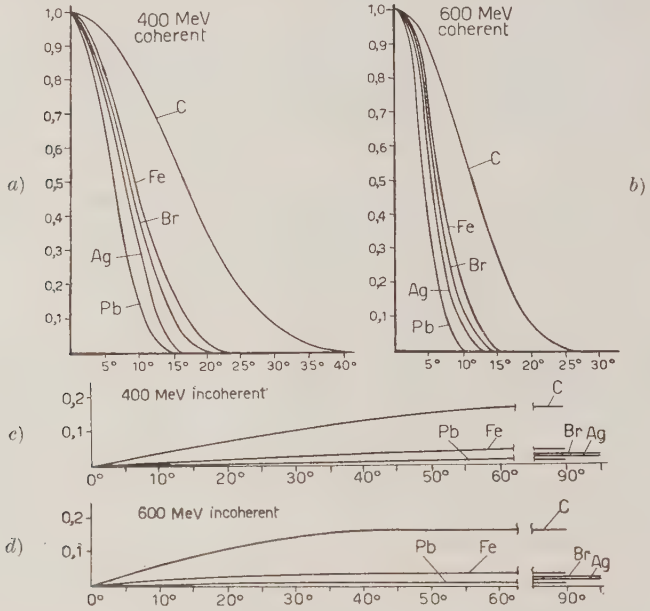

Fig. 1.

can arise from an incoherent scattering of an electromagnetic nature. There exists an incoherent contribution, certainly, and it is preponderant with respect to the coherent contribution at large angles; it is however a negligible contribution in the heavier nuclei and its manner of dependence upon Z is the opposite of that which would be required to explain a larger anomalous cross-section in heavy nuclei than in light nuclei. In Fig. 1 some curves are shown of the incoherent form factor for those elements which concern the present experiments, together with some evaluations of the coherent form factor made with Born's approximation and assuming a uniform charge distribution. It can be verified how the incoherent contribution in the heavy nuclei becomes very small with respect to the coherent contribution except at very large angles.

Concerning again the possibility of models containing components of large momenta, only to the end of obtaining an indication, as an extreme case and

although such models are now out of date, an α-particle model has been considered here which can contain components of indefinitely large momenta if the radius of the α-particles is chosen to be very small with respect to the radius of the nucleus. The case is considered of Fe^{56} which because of the way in which it was thought to be constructed on an α-particle model (one central α-particle plus 12 α-particles at the vertices of an icosahedron [9]), seems a priori to be the most likely case to give large cross-sections. If for the nucleons which constitute the α-particle, oscillatory wave functions are chosen of the form $\frac{N}{A}\exp\left[-\frac{1}{2}(\boldsymbol{R}-\boldsymbol{r}_i)^2/a^2\right]$ where $\boldsymbol{r}_i$ is the baricentre of the i-th α-particle, it is found that the form factor for the coherent scattering $\mathcal{E}$ can be represented, to a good approximation, by an expression of the type

$$Z^{-2}\exp\left[-\frac{1}{2}(ka)^2\right]\left[\frac{2}{Z}+\left(1-\frac{2}{Z}\right)\frac{\sin\alpha kr_0}{\alpha kr_0}\right]^2. \tag{3}$$

r_0 being the usual constant in $R=r_0A^{1/3}$, α a factor depending on the definition adopted here for the radius of the nucleus (square root of the mean square of the radius of the charge distribution of the peripheral α-particles)

$$\alpha=A^{1/3}\left[1-\frac{3}{2}\left(\frac{a}{r_0}\right)^2A^{-2/3}\right]^{1/2}.$$

In the expansion of the square in (3) the term $(2/Z)^2$ represents the scattering of the central α-particle, the other quadratic term represents the scattering of the α-particles at the vertices of the icosahedron, and in addition there is the interference term; the factor $\exp\left[-\frac{1}{2}(ka)^2\right]$ takes account of the finite dimensions of the α-particles. In fact for $a\to 0$, i.e. the case in which the radius of the α-particles is very small with respect to the radius of the nucleus, the term representing the scattering of the central α-particle gives a contribution to the form factor $\mathcal{E}$ constant at all angles however large; this contribution however is extremely small ($-1/13^2=1/169$), and indeed for angles at which the cross-section has values still appreciable the interference term is more important. In any case a scattering is obtained which is always less than that obtained simply with a uniform charge distribution (see Fig. 2: the curve 1 refers to this last model).

Another point to be examined, and which naturally has yet to be correlated with the presence of components of large momenta is the dependence of the cross-sections on the radius of the nucleus. The necessity of discussing this point arises also from the results of the recent experiments of Rainwater [10] on X-rays emitted in transitions of mesic atoms, which have allowed

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[9] Rosenfeld: *Nuclear Forces*, **13**, 13 (1937); Wefelmeyer: *Naturwissenschaft*, **25**, 525 (1937); *Zeits. f. Phys.*, **107**, 332 (1937).

[10] Not published.

a measurement of the radii for electromagnetic interaction of nuclei, thus suggesting the possibility that r_0 assumes a value less than $1.4 \cdot 10^{-13}$ cm. The results of previous experiments on the scattering of electrons by nuclei made by LYMAN, HANSON and SCOTT (1951) ([11]) tend in the same direction. These suggested a reduction of about 20% for r_0. It does not seem however to be deducible that a reduction of r_0 within these limits can obviate the discrepancy in the scattering of μ-mesons, which would need a reduction by a factor $\simeq 10$ (see Fig. 3: the calculations were made using Born's approximation).

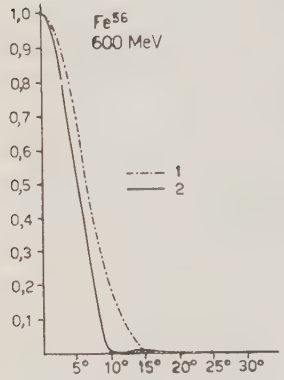

Fig. 2.

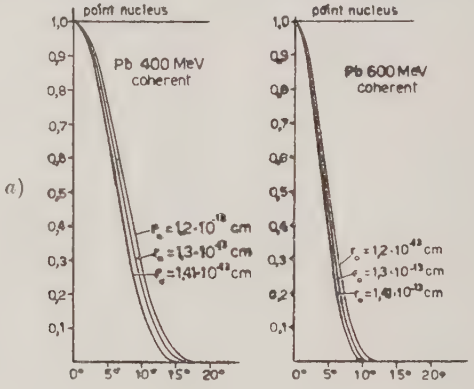

Fig. 3.

A point which still need accurate investigation concerns the use of Born's approximation, especially where it concerns the coherent part. For heavy nuclei Born's approximation may not be valid. At the energies considered a large number of phases presumably take part. For very heavy nuclei many of these phases are presumably large and, especially at large angles where Born's approximation has less validity, the comparison with a classic solution can be significant, although it is still outside its limit of validity.

The spin can increase the incoherent contribution especially at large angles. In the coherent contribution one can foresee a factor $\simeq [1 - \beta^2 \sin^2(\theta/2)]$: so the formulae without spin already constitute an increase.

A discussion on the contribution of virtual π-mesons is contained in the work of AMALDI *et al.* ([8]). Here the aim is only to make use of the closure approximation to obtain more general conclusions to be applied to heavy nuclei. Although the contribution of the neutrons is such as to make itself felt more in heavy nuclei than in light nuclei because of the increase of the ratio N/Z,

([11]) E. M. LYMAN, A. O. HANSON and M. B. SCOTT: *Phys. Rev.*, **84**, 626 (1951).

the calculated contribution is still very small, although this point needs hypotheses, presumably not valid, on the nature of the meson field responsible for the interaction between the nucleons.

As a final point there remains to be considered the contribution due to the anomalous magnetic moments. As regards the anomalous magnetic moments of the nucleons, in scattering experiments, this contribution is presumably still small. However, the contribution which can arise from the existence of an anomalous magnetic moment for the μ-meson is not considered here.

Particular considerations and details of calculation.

A Dirac equation of second order is assumed to hold for the μ-meson

$$[(E - e\varphi)^2 - c^2\boldsymbol{p}^2 - m_0^2c^4 + ie\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{E}]\,\psi = 0\,. \tag{4}$$

where E is the total energy, $\boldsymbol{p}$ the momentum, m_0 the mass, $e\varphi = V$ the potential, $\boldsymbol{\alpha}$ the usual Dirac matrix and $\boldsymbol{E}$ the electric field; for the nucleons on the other hand a non-relativistic description is adopted. If in (4) the spin and the φ^2 term are ignored, the usual non-relativistic equation

$$[\nabla^2 + K^2 - (2m/\hbar^2)V]\,\psi = 0,$$

is obtained, where however, instead of m, its relativistic expression $m_0(1-\beta^2)^{-1/2}$ is considered to be substituted. It is clear that the same result is reached if, instead of starting from (4), one starts from a relativistic Schrödinger equation. The formulae used from now on will differ from the corresponding non-relativistic formulae only by the substitution of $m_0(1-\beta^2)^{-1/2}$ for the mass m.

The cross-section relative to the scattering of the incident μ-meson in $\mathrm{d}\omega$ with excitation of the nucleus from its ground state to the state m is given, in Born's approximation, by

$$\frac{\mathrm{d}\sigma_{0m}}{\mathrm{d}\omega} = \frac{2\pi}{\hbar}\,\varrho(E)\,\frac{\Omega m}{p_0}\,|H_{0m}|^2$$

$$H_{0m} = \frac{1}{\Omega}\int \Psi_0^* \sum_{i=1}^{z} \frac{e^2}{\boldsymbol{R}_i - \boldsymbol{r}} - \Psi_m \exp\left[i(\boldsymbol{K} - \boldsymbol{K}_0)\boldsymbol{r}\right] \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{R}_1 \ldots \mathrm{d}\boldsymbol{R}_z\,.$$

(Ω is the normalisation volume, $\boldsymbol{K}_0 = \boldsymbol{p}_0/\hbar$ the initial wave-vector, $\boldsymbol{K} = \boldsymbol{p}/\hbar$ the final wave-vector of the μ-meson, $\boldsymbol{r}$ the position vector of the meson, $\boldsymbol{R}_i$ the position vector of the i-th proton, $\varrho(E)$ the density of the final states, and Ψ are wave functions of the nucleus). The calculations are made in the laboratory

system which practically coincides with the center of mass system. The preceding formulae can also be written.

$$(5)\qquad \frac{\mathrm{d}\sigma_{0m}}{\mathrm{d}\omega} = \frac{4e^4}{c^4}\frac{E_0}{p_0}\frac{pE}{q^4}\,|A_{0m}|^2 \qquad A_{0m} = \int \Psi_0^* A \Psi_m \mathrm{d}\tau\,, \quad A = \sum_{i=1}^{Z} \exp\,[i\boldsymbol{k}\cdot\boldsymbol{R}_i]\,,$$

where $\boldsymbol{q} = \hbar\boldsymbol{k} = \hbar\boldsymbol{K} - \hbar\boldsymbol{K}_0$.

Let $\eta = p - p_0$ be the diminution of momentum of the meson after collision. η is assumed small, which is equivalent to limiting this theory to high energies for the incident μ-meson, and pE/q^4 expanded in a power series in η

$$\frac{pE}{q^4} = \frac{E_0}{16p_0^3\sin^2(\theta/2)}\left[1 + \left(\frac{\mu c^2}{E_0}\right)^2\frac{\eta}{p_0} + \right.$$

$$\left. + \frac{(1-\cos\theta)[1 + c^2p_0^2/E_0^2 - c^4p_0^4/E_0^4] - 1}{1-\cos\theta}\left(\frac{\eta}{p_0}\right)^2 + \ldots\right].$$

The coefficient of the first order term is already small on its own account for energies E_0 of the incident meson sufficiently greater than its rest-energy, which is of the order of 107 MeV. In the following the zero order term of the previous expansion is substituted for pE/q^4. The second order term shows that the approximation no longer holds for $\theta = 0$.

To the order of the approximation made it will be supposed that $k_\perp = 2K_0\sin(\theta/2)$ always, and the sum will be extended over all the matrix elements to obtain the total cross-section (i.e. the limitation due to energy conservation will not be taken into account, which would require the sum to be terminated at a finite number of matrix-elements, since presumably the new terms thus introduced are very small)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} \cong \frac{1}{4}\frac{e^4}{c^4}\frac{E_0^2}{p_0^4\sin^4(\theta/2)}\sum_{m=0}^{\infty} A_{0m}^* A_{0m}$$

$$\sum_{m=0}^{\infty} A_{0m}A_{0m} = \int\!\!\int \Psi_0^*(1)A(1)A^*(2)\Psi_0(2)\,\mathrm{d}(1)\,\mathrm{d}(2)\sum_m \Psi_m(1)\Psi_m^*(2) = \int \Psi_0^*[AA^*]\Psi_0\,\mathrm{d}\tau\,.$$

It must be noted at this point that the approximation made allows one to avoid hypotheses on the excited states of the nucleus. Now

$$(6)\qquad AA^* = \sum_{i=1}^{Z}\sum_{k=1}^{Z}\exp\,[i\boldsymbol{k}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_k)] = Z + 2\sum_{i>k}^{Z}\sum\cos\boldsymbol{k}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_k)\,,$$

so that

$$(1\text{-}bis)\quad \mathcal{C} + \mathcal{J} = Z^{-2}\int \Psi_0^*[AA^*]\Psi_0\,\mathrm{d}\tau =$$

$$= Z^{-2}\left[Z + 2\int \Phi_0^*\sum_{i>k}^{Z}\sum\cos\boldsymbol{k}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_k)\Phi_0\,\mathrm{d}\tau\right],$$

having integrated over the coordinates of the neutrons and so substituted for Ψ_0 a certain Φ_0.

The integral on the right hand side has the same form as the integral giving the mean value of a fictitious potential energy due to interactions

$$V(i, k) = \cos \boldsymbol{k} \cdot (\boldsymbol{R}_i - \boldsymbol{R}_k) ,$$

between the pairs of nucleons. Therefore direct terms proportional to Z^2 and exchange terms proportional to Z are to be expected. The former correspond to the coherent scattering, the latter to the incoherent scattering.

If one assumes for the nucleus an independent particle model with a wave function antisymmetrical with regard to all neutrons and protons, one has, as is well known,

$$\int \Phi_0^* \sum_{i>k}^{Z} \sum V(i, k) \Phi_0 \, \mathrm{d}\tau =$$

$$= \frac{1}{2} \sum_{i=1}^{Z} \sum_{k=1}^{Z} \left[(\psi_i^*(1) \psi_k^*(2) \mid V(1, 2) \mid \psi_i(1) \psi_k(2)) - (\psi_i^*(1) \psi_k^*(2) \mid V(1, 2) \mid \psi_k(1) \psi_i(2)) \right],$$

where ψ_i are the individual wave functions of the single protons which are assumed of the form $\varphi_i \chi_i$ where φ_i is the spatial part, χ_i the spin part.

Then one obtains ($Z_+ = \frac{1}{2}Z$)

$$\int \Phi_0^* A A^* \Phi_0 \, \mathrm{d}\tau = Z + 4 \sum_{i=1}^{Z_+} \sum_{k=1}^{Z_+} (\varphi_i^*(1) \varphi_k^*(2) \mid V(1, 2) \; \varphi_i(1) \varphi_k(2)) -$$

$$- 2 \sum_{i=1}^{Z_+} \sum_{k=1}^{Z_+} (\varphi_i^*(1) \varphi_k^*(2) \mid V(1, 2) \mid \varphi_k(1) \varphi_i(2)) .$$

The expressions

$$\lambda_1 = \sum_{i=1}^{Z_+} |\varphi_i(1)|^2 , \qquad \varrho = \sum_{i=1}^{Z_+} \varphi_i^*(1) \varphi_i(2) ,$$

are now introduced where the sums are taken over all the first Z_+ wave functions of the central potential considered. Separating the coherent from the incoherent scattering with the criterion already explained the following expressions are obtained:

$$(7) \quad \begin{cases} \left(\dfrac{\mathrm{d}\sigma}{\mathrm{d}\omega} \right)_{\text{coer}} = \dfrac{1}{4} \dfrac{e^4}{c^4} \dfrac{E_0^2}{p_0^4 \sin^4(\theta/2)} \left| \int \lambda \exp [i \boldsymbol{k} \cdot \boldsymbol{R}] \, \mathrm{d}\boldsymbol{R} \right|^2 , \\ \left(\dfrac{\mathrm{d}\sigma}{\mathrm{d}\omega} \right)_{\text{inc}} = \dfrac{1}{4} \dfrac{e^4}{c^4} \dfrac{E_0^2}{p_0^4 \sin^4(\theta/2)} \left[Z - 2 \iint \mathrm{d}(1) \, \mathrm{d}(2) \cos \boldsymbol{k} \cdot (\boldsymbol{R}_1 - \boldsymbol{R}_2) \mid \varrho(1, 2) \mid^2 \right] . \end{cases}$$

The coherent scattering depends on the wave function of the single protons only through the expression

$$2\lambda_1 = 2 \sum_{i=1}^{Z_+} |\varphi_i(1)|^2 ,$$

which expresses the total density of the protons, and so is proportional to the Coulomb charge distribution in the nucleus. Therefore the coherent scattering is calculable only on the basis of hypotheses on the charge distribution. The curves given in figs. 1 and 3 refer to a uniform charge distribution.

For the incoherent scattering the dependence on the single wave functions φ_i is contained in

$$\varrho = \sum^{Z+} \varphi_i^*(1)\varphi_i(2) .$$

This expression has values appreciably different from zero only if the two arguments $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are separated by a distance of the order of r_0 or less: on the other hand over regions of dimension r_0 it will in general be legitimate to approximate the φ_i with plane waves.

In conclusion, while for the coherent scattering it is sufficient to make hypotheses on the form of the charge distribution, for the incoherent scattering a strong dependence on the particular choice of the φ_i is not to be expected.

If the φ_i are now replaced with plane waves the function $\varrho(1, 2)$ can be evaluated substituting integrals for the sums over states

$$\varrho(R_{12}) = \frac{1}{2} \frac{\sin cR_{12} - cR_{12} \cos cR_{12}}{\pi^2 R_{12}^3} , \qquad c = \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_0} \left(\frac{Z}{A}\right)^{1/3} ,$$

(c is the maximum wave number of the protons in the nucleus).

It is now necessary to evaluate the integral

$$(8) \qquad \iint \mathrm{d}(1)\,\mathrm{d}(2) \cos \boldsymbol{k} \cdot (\boldsymbol{R}_1 - \boldsymbol{R}_2) \varrho^2(R_{12}) = \int \mathrm{d}\boldsymbol{R}_2 \int \mathrm{d}\boldsymbol{R}_{12} \cos \boldsymbol{k} \cdot \boldsymbol{R}_{12}\, \varrho^2(R_{12}) ,$$

where the substitution $\boldsymbol{R}_{12} = \boldsymbol{R}_1 - \boldsymbol{R}_2$ has been made.

The integral (8) is taken twice throughout the volume occupied by the nucleus. For large angles it is convenient to make use of the fact that the function $\varrho(R_{12})$ tends rapidly to zero for $R_{12} \to \infty$, and thus to take the integration with respect to R_{12} from 0 to ∞. In this way one is sure to make an error at small angles. In the interpretation used here it can be said that this approximation will be sufficiently correct only where $V(i, k)$ represents a short range potential; and this is true at large angles because of the oscillatory nature of $V(i, k)$ but it is not true at small angles.

In this approximation the following expression for the integral (8) is obtained

$$(9) \qquad \frac{c^4}{k} \frac{V}{\pi^3} \int_0^\infty x^{-5}\,\mathrm{d}x\,(\sin x - x \cos x)^2 \sin bx , \qquad b = \frac{k}{c} ,$$

V = volume of the nucleus.

The integral

$$(10) \qquad \int_0^\infty x^{-5}\,\mathrm{d}x \sin bx\,(\sin x - x \cos x)^2,$$

is a function of b identically zero for $b \geqslant 2$.

For $0 < b < 2$ it is given by

$$\frac{\pi}{24}\left[\frac{1}{4}b^4 - 3b^2 + 4b\right], \tag{11}$$

and this expression, and its first derivative are equal to zero for $b = 2$.

Since the evaluation of (10) is not immediate it is perhaps instructive to give a brief reference to it so as to show the cause of its singular behaviour, especially in view of the clear physical interpretation of the result which will be shown in the following. The integral is written in the form $f(x)$+complex conjugate.

$$f(x) = -\frac{1}{8}x^{-5}\{2(1+x^2)i\exp[ibx] + i(x^2-1)\exp[i(b+2)x] + \tag{12}$$

$$+ i(x^2-1)\exp[i(b-2)x] + 2x\exp[i(b-2)x] - 2x\exp[i(b+2)x]\}.$$

This expression is meant to be extended over the complex plane. First suppose that $b > 2$ and consider the path of integration in Fig. 4. Since the integral over Γ at the limit (for $\varrho \to 0$, $R \to \infty$) is zero, the integral (10) becomes equal to the limit of the integral

$$(\overleftarrow{\gamma})\int f(z)\,dz\,.$$

This integral is necessarily equal to $i\pi\times$(coefficient of x^{-1} in the Laurent expansion of $f(x)$ about the origin), since the convergence of (10) guarantees that the coefficients of the other negative powers must be zero. It is immediately verified from (12) that this coefficient is zero, so that for $b > 2$ the integral (10) is identically zero. The difference in behaviour in the case $0 < b < 2$ arises from the fact that now the integral over Γ no longer tends to zero. It is sufficient however to break up $f(x)$ into the sum of two terms, $\varphi(x)$ comprising the positive exponentials, and $\psi(x)$ comprising the negative exponentials; the former path of integration is used for the first term, for the second term the one shown below is used (Fig. 5).

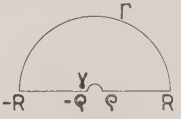

Fig. 4.

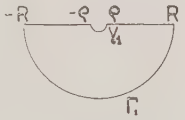

Fig. 5.

Since the integrals over Γ and Γ_1 both go to zero (Jordan's lemma) the integral (10) can be evaluated as the limit of the sum

$$(\overleftarrow{\gamma})\int\varphi(z)\,dz + (\gamma_1)\int\psi(z)\,dz = i\pi\times(\text{coefficient of } x^{-1} \text{ of } \varphi) -$$

$$- i\pi\times(\text{coefficient of } x^{-1} \text{ of } \psi) = \text{expression (11)}.$$

The physical situation corresponding to this result is illustrated in Fig. 6. Firstly it is noted how in this nuclear model the only correlation between the nucleons is that imposed by the exclusion principle. Corresponding to a predeterminated momentum $\boldsymbol{p}_r = \hbar\boldsymbol{k}$ which is given by the μ-meson during the collision only those protons which have momenta in the zone of the Fermi sphere indicated by 1 can contribute to the incoherent scattering. This zone contains those momenta which, always being contained in the Fermi sphere,

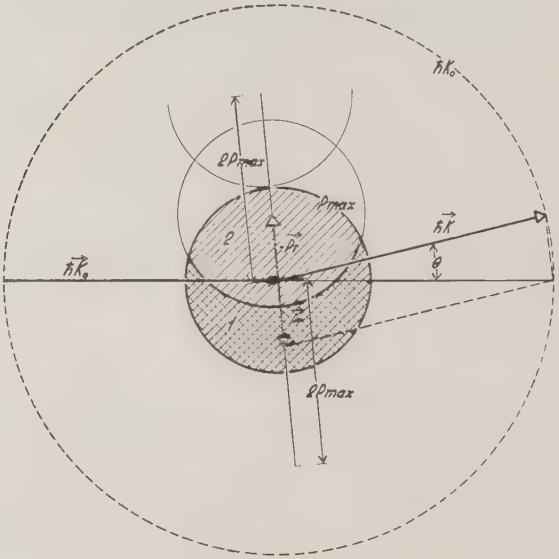

Fig. 6.

are however external to an identical sphere whose centre is displaced by $-\hbar\boldsymbol{k}$ with respect to the centre of the Fermi sphere. The protons which are in this zone have, after the collision, a momentum certainly greater than p_{max}. The protons in the zone 2 of the Fermi sphere have, on the other hand, a momentum less than p_{max} after collision. Therefore the protons initially situated in zone 2 cannot contribute to the scattering because such a transition would be prohibited by the exclusion principle. All the protons will scatter as Z distinct scattering centres as soon as the momentum given up by the incident μ-meson becomes greater than twice the maximum momentum they can have in the nucleus: only then indeed, as is evident from the figure, is the zone 1 identical with the whole Fermi sphere. This is in fact quite clear as soon as it is realised that the worst case possible is that of a proton with a momentum parallel to the recoil momentum $\boldsymbol{p}_r$ which it receives, but in the opposite direction. So that even this transition be not forbidden by the exclusion principle it is necessary and sufficient that the recoil momentum p_r given to the proton struck be at least twice the momentum p_{max}. This condition is exactly equivalent to the condition $b \geqslant 2$; in fact $b \geqslant 2$ means that

$k \geqslant 2c$, i.e. $p_r \geqslant 2p_{\max}$. When this condition is satisfied all the single protons scatter as Z independent centres, and this is the physical content of the preceding analytical result.

Making use of expression (11), the following result is obtained for the differential cross-section for incoherent scattering:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)_{\mathrm{inc}} = \frac{1}{4}\frac{e^4}{c^4}\frac{E_0^2 Z^2}{p_0^4 \sin^4(\theta/2)}\,\mathcal{J}\,,$$

with

$$\text{(12)} \qquad \begin{cases} \mathcal{J} = Z^{-1}\left\{-\frac{1}{2}\left[\frac{\sin(\theta/2)}{\sin(\bar{\theta}/2)}\right]^3 + \frac{3}{2}\frac{\sin(\theta/2)}{\sin(\bar{\theta}/2)}\right\} & \text{for} \quad \theta \leqslant \bar{\theta}\,, \\ \mathcal{J} = Z^{-1} & \text{for} \quad \theta \geqslant \bar{\theta}. \end{cases}$$

where $\bar{\theta}$ is defined by the relation

$$\text{(13)} \qquad 2\sin(\bar{\theta}/2) = 2(9\pi/4)^{1/3}(Z/A)^{1/3}(\lambda\!\!\!^{-}/r_0)\,.$$

Some curves relative to the factor $\mathcal{J}$ have already been given in Fig. 1.

It is to be noted that it can in fact be stated that for $\theta > \bar{\theta}$ the nucleus scatters exactly as Z separate protons. In fact for $Z = N$ (13) gives $2\sin(\bar{\theta}/2) \cong 3{,}04(\lambda\!\!\!^{-}/r_0)$ while the maximum angle of coherent scattering, $\theta_{\max}$, in a model with a uniform charge distribution is given by $kR_0 = kr_0A^{1/3} \cong 4{,}49$, i.e. by the equation $2\sin(\theta_{\max}/2) = 4{,}49A^{-1/3}(\lambda\!\!\!^{-}/r_0)$ and for the values of A considered one has effectively $\bar{\theta} > \theta_{\max}$.

The inequality $\mathcal{J} \leqslant Z^{-1}$ can be demonstrated in more general hypotheses. Starting from (7), consider the expression in square brackets which can be written in the form.

$$\text{(15)} \qquad Z - \frac{1}{2}\sum_{i=1}^{Z}\sum_{k=1}^{Z}\left(\psi_i^*(1)\psi_k^*(2)\,\exp\left[i\cdot\boldsymbol{k}(\boldsymbol{R}_1 - \boldsymbol{R}_2)\right]\,\psi_k(1)\psi_i(2)\right) -$$

$$- \frac{1}{2}\sum_{i=1}^{Z}\sum_{k=1}^{Z}\left(\psi_i^*(1)\psi_k^*(2)\,|\exp\left[-i\boldsymbol{k}\cdot(\boldsymbol{R}_1 - \boldsymbol{R}_2)\right]|\,\psi_k(1)\psi_i(2)\right),$$

that is

$$\text{(16)} \qquad Z - \sum_{i=1}^{Z}\sum_{k=1}^{Z}|(\psi_i^*|\exp\left[i\boldsymbol{k}\cdot\boldsymbol{R}\right]|\psi_k)|^2,$$

if the indices i and k and then the indices 1 and 2 are interchanged in the second sum in (15). This expression is certainly always less than Z, from which the preceding inequality is obtained remembering the definition of $\mathcal{J}$.

It is clear that the only hypothesis contained in the preceding result is the hypothesis that the description of the nucleus by means of an independent particle model is adequate; and its physical meaning in the case of a Fermi model has already been explained above.

For light nuclei sinple analytical expressions can be obtained, by the closure approximation, for the form factor of the incoherent scattering, making use of a shell model with oscillators. The case of C^{12} will be considered here as an example, although more complicated cases could also be calculated. The result will be compared with the result contained in the work of AMALDI *et al.*, where, although essentially under the same hypotheses, the closure approximation is not used. Thus the validity of the closure approximation will be verified.

The excessive degeneracy of the levels of the oscillator does not in general allow a unique choice of the individual wave functions with which to construct the wave function of the nucleus in its ground state. With the object of partly avoiding this degeneracy a spin-orbit coupling is here supposed to be present, as in the model of JENSEN [12] and MAYER [13]. It is well known that in this model a spin-orbit coupling is assumed, which depresses the energy levels of the states of larger total angular momenta among states having the same orbital angular momentum. The carbon C^{12} is thus described by a configuration of the type

$$[(1s_{1/2})^2(1p_{3/2})^4;\ (1s_{1/2})^2,\ (1p_{3/2})^4]\,.$$

In this notation, as is usual in nuclear spectroscopy and in contrast to the notation used in atomic spectroscopy, the number preceding the symbol of angular momentum, minus 1, gives directly the number of radial nodes.

The individual wave functions are of the form

$$(17)\qquad \psi\left(n, l, j = l + \frac{1}{2}, m\right) =$$

$$= \frac{R_{n,l}(r)}{r}\left[\frac{(l + m + 1/2)^{1/2}}{(2l+1)^{1/2}}\,\alpha Y_l^{m-1/2} + \frac{(l - m + 1/2)^{1/2}}{(2l+1)^{1/2}}\,\beta Y_l^{m+1/2}\right],$$

or

$$\psi\left(n, l, j = l - \frac{1}{2}, m\right) =$$

$$= \frac{R_{n,l}(r)}{r}\left[-\frac{(l - m + 1/2)^{1/2}}{(2l+1)^{1/2}}\,\alpha Y_l^{m-1/2} + \frac{(l + m + 1/2)^{1/2}}{(2l+1)^{1/2}}\,\beta Y_l^{m+1/2}\right],$$

according to whether, in the states considered, the spin is added to or substracted from the orbital angular momentum l. The radial parts of the wave functions are given by expressions of the form (n here gives the number of radial nodes directly)

$$R_{n,l}(r) = N_{n,l}\exp\left[-(\nu/2)r^2\right]r^{l+1}V_{n,l}(r)\,,$$

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[12] O. HAXEL, J. H. O. JENSEN and H. E. SUESS: *Zeits. f. Phys.*, **128**, 295 (1950).
[13] M. G. MAYER: *Phys. Rev.*, **75**, 1969 (1949).

where $v = \omega m/\hbar$, $\frac{1}{2} m\omega^2 r^2$ is the potential, $N_{n,l}$ is a normalising factor, and $V_{n,l}(r)$ is an associated Laguerre polynomial

$$V_{n,l}(r) = L^{l+1/2}_{n+l+1/2}(vr^2) .$$

The calculation is made easier if the wave functions (17) are expressed in cartesian coordinates. The first six wave functions, which are the only ones of interest here, can thus be written in the form

$$\begin{aligned}
\psi_1 &= \varphi(0, 0, 0)\alpha , \\
\psi_2 &= \varphi(0, 0, 0)\beta , \\
\psi_3 &= -2^{-1/2}[\varphi(0, 1, 0) + i\varphi(1, 0, 0)]\alpha , \\
\psi_4 &= 3^{-1/2}\{2^{1/2}\varphi(0, 0, 1)\alpha - 2^{-1/2}[\varphi(0, 1, 0) + i\varphi(1, 0, 0)]\beta\} , \\
\psi_5 &= 3^{-1/2}\{2^{1/2}\varphi(0, 0, 1)\beta + 2^{-1/2}[\varphi(0, 1, 0) - i\varphi(1, 0, 0)]\alpha\} , \\
\psi_6 &= 2^{1/2}[\varphi(0, 1, 0) - i\varphi(1, 0, 0)]\beta ,
\end{aligned}$$

where

$$\varphi(m_1, m_2, m_3) = \varphi_{m_1}(x)\varphi_{m_2}(y)\varphi_{m_3}(z) ,$$

$\varphi_{m_k}(x)$ being, apart from a normalising factor the product of an exponential $\exp[-\frac{1}{2}vx^2]$ and a Hermite polynomial of order m_k. The integrals appearing in (16) can thus be reduced to integrals

$$(\varphi(m_1, m_2, m_3)|\exp[i\boldsymbol{kR}]\ \varphi(m_1', m_2', m_3')) = (\varphi_{m_1}|\exp[ikx]\ \varphi_{m_1'})\delta_{m_2,m_2'}\delta_{m_3,m_3'} ,$$

which can be evaluated immediately. The expression thus obtained for the form factor of the incoherent scattering $\mathcal{J}$, in the case of C^{12} is the following

$$\mathcal{J} = (1/6) - e^{-u}[(2/81)u^2 + (1/6)] \qquad u = \tfrac{1}{2}(ka)^2 a^2 = \nu^{-1} . \tag{18}$$

For a the value here assumed is that chosen by AMALDI *et al.*, $a = 2.04 \cdot 10^{-13}$ cm which arises from having defined the radius R_0 of the nucleus by means of the relation $R_0^2 = \overline{R^2}$, $\overline{R^2}$ being the mean square radius of the charge distribution of the last proton. In Fig. 7 the curves (1) are taken from the work of AMALDI *et al.*, where they were obtained with a shell model with oscillators, without however, a spin orbit coupling, and without using the closure approximation, but using the same hypotheses used in this work, and the same values of all the parameters and constant assumed here. The curves (2) refer to formula (18) which was obtained by closure approximation, with a shell model with oscillators and with spin-orbit coupling to avoid the degeneracy of levels. A slight difference between the two curves can be seen at large angles. This difference cannot arise from the closure approximation since it is known that the errors come at small angles with this approximation. It can be recognised

that the difference is essentially due to the different criterion used in filling the levels. The behaviour of curves (2), which at large angles seem to diverge very slightly from the limiting value Z^{-1}, which however is to be expected physically, is presumably to be related to the fact that in the work of AMALDI *et al.* all the levels corresponding to a certain energy are supposed to be uniformly filled and then the cross-section is evaluated; instead of evaluating

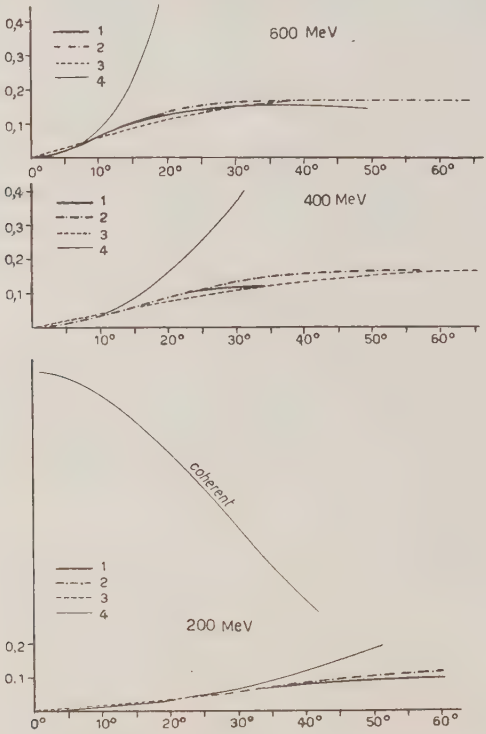

Fig. 7.

first the cross-sections corresponding to the different ways of filling the levels and only then taking the mean cross-section. The first procedure, although quite reasonable in the work of AMALDI *et al.*, would give appreciable errors when the closure approximation is used and so it has been found better here to assume a spin-orbit coupling to eliminate the degeneracy. The curves (3) were obtained using formula (12) furnished by the statistical model. The difference at small angles arises essentially from having taken the integral (8) from 0 to ∞ and from the particular distribution of the momenta typical of a Fermi model. If account is taken of the fact that the curves 2 and 3 refer to two different nuclear models, and that a statistical model for a light nucleus like carbon has certainly little validity, and if one finally considers that at small angles very rigorous calculations are not necessary since at these angles

the incoherent contribution is negligible with respect to the coherent, the agreement must be considered as satisfactory. The curves 4 were obtained by stopping at the terms in k^2.

It is desirable to discuss here the contribution which the virtual π-mesons can make to the scattering. The problem has already been discussed by AMALDI *et al.* ([8]). However, using the closure approximation, always from the point of view of a phenomenological formulation of the problem, simple and general results can be obtained. From a theoretical point of view, the problem has been dealt with by ROSENBLÜTH ([14]) and CORINALDESI ([15]). To take account of the contribution of the virtual π-mesons in a perturbation scheme means to consider the contribution of graphs in which the proton, passing into the neutron state, emits a virtual meson which absorbs the virtual photon, and is then reabsorbed by the neutron, and graphs in which the meson is emitted and reabsorbed before or after the scattering. The result obtained ([14]) has a simple physical interpretation and leads to a conception of the charge of the proton as being extended over a zone whose dimensions are of the order of the Compton wave-length of the π-meson, so that in distant collisions all the charge of the proton is effective, while in nearby collisions only a fraction of it is effective. If these considerations are applied to the protons in a nucleus, analogous considerations should be applied to the neutrons. A reverse situation can be imagined for a neutron. That is, in the nearby collisions there acts only the actual charge zero of the bare neutron while in the distant collisions the charge carried by the meson cloud intervenes. Taking all together it can be expected that in a nucleus with an equal number of neutrons and protons the two effects, both for distant and nearby collisions, balance each other for the coherent scattering, and a different result can be due only to the presence of an excess of neutrons. The discussion of the incoherent contribution however, because of the absence of interference terms, gives rise in general to different results. It is clear that the validity of the preceding conclusions, just as for the conclusions of ROSENBLÜTH, is correlated with the validity of a perturbation scheme for the treatment of the interaction between π-mesons and nucleons, a scheme which on the contrary has presumably little validity; the only thing that can be concluded is that the modifications obtained in this way for the cross-sections, even for nuclei with a strong isotopic excess, are so small as to let one think that a better formulation of the problem would not lead to any appreciable modification in the results. As in the work of AMALDI *et al.*, we begin first by defining the following « Ansatz » concerning the charge density of a proton and a neutron: $\varrho_p = t\varrho_P + (1-t)\varrho_\pi$ and $\varrho_n = (1-t)(\varrho_P - \varrho_\pi)$ where t is the fraction of time in which each nucleon is in its pure state of proton or neutron, ϱ_P the charge density relative to the pure state of proton, which for all the energies considered here of the incident μ-meson can be considered to be represented by a δ-function, and ϱ_π the charge distribution relative to the virtual π-meson, which extends over a zone whose dimensions are of the order of the Compton wave-length of the meson.

([14]) M. N. ROSENBLÜTH: *Phys. Rev.*, **79**, 615 (1950).
([15]) E. CORINALDESI: *Nuovo Cimento*, **7**, 61 (1951).

In the evaluation of the matrix element $H_{0m} = \Omega^{-1}$

$$\int \Psi_0^* \Psi_m \exp[i\boldsymbol{k}\boldsymbol{r}] \sum u_i \, \mathrm{d}\boldsymbol{r} \, \mathrm{d}\tau ,$$

where u_i is the Coulomb potential due to the i-th nucleon, the equation $\Delta u_i = -4\pi e^2 \varrho_i$ and the identity $\Delta \exp[i\boldsymbol{k}\cdot\boldsymbol{r}] = -k^2 \exp[i\boldsymbol{k}\cdot\boldsymbol{r}]$ are used. Integrating by parts, the expression

$$\int \mathrm{d}\boldsymbol{r} \exp[i\boldsymbol{k}\cdot\boldsymbol{r}] \sum u_i$$

becomes

$$(4\pi e^2/k^2) \left\{ \sum_{i=1}^{Z} \left[t + (1-t)\, \mathcal{F}(k) \right] \exp[i\boldsymbol{k}\cdot\boldsymbol{R}_i] + \sum_{r=1}^{N} (1 \quad t) \left[1 \quad \mathcal{F}(k) \right] \exp[i\boldsymbol{k}\cdot\boldsymbol{R}_r] \right\},$$

since integrals of the type

$$\int \mathrm{d}\boldsymbol{r} \exp[i\boldsymbol{k}\cdot\boldsymbol{r}] \varrho_p(i)$$

give $\exp[i\boldsymbol{k}\cdot\boldsymbol{R}_i]$ because of the supposition that the charge distribution of the pure nucleon can be considered puntiform; while the integrals of the type

$$\int \mathrm{d}\boldsymbol{r} \exp[i\boldsymbol{k}\cdot\boldsymbol{r}] \varrho_\pi(i)$$

can be written in the form $\exp[i\boldsymbol{k}\cdot\boldsymbol{R}_i]\mathcal{F}(k)$, where

$$\mathcal{F}(k) = (4\pi/k) \int_0^\infty v \, \mathrm{d}v \sin kv \varrho_\pi(v)$$

is a form factor depending on the momentum k given up by the meson during the collision, and which takes account exactly of the different apparent values of the charge of the nucleon according to whether one deals with nearby collisions (large values of k) or distant collision (small values of k). If one supposes for example that ϱ_π is gaussian, $\mathcal{F}(k)$ will also be a gaussian (see AMALDI *et al.* [8]).

We follow here the interpretation of ROSENBLÜTH and SCHIFF choosing $\varrho_\pi = (x/2\pi)v^{-2} \exp[-2xv]$ to which corresponds $\mathcal{F}(k) = (2x/k)$ arctg $(k/2x)$. If one puts $\mathcal{P}(k) = t + (1-t)\mathcal{F}(k)$ and $\mathcal{N}(k) = (1-t)[1-\mathcal{F}(k)]$ it is easy to verify that the operator A which occurs in (5) must now be replaced by the operator

$$\mathcal{P}(k) \sum_{i=1}^{Z} \exp[i\boldsymbol{k}\cdot\boldsymbol{R}_i] + \mathcal{N}(k) \sum_{r=1}^{N} \exp[i\boldsymbol{k}\cdot\boldsymbol{R}_r] .$$

This corresponds to substituting for Z in (6) the expression $\mathcal{P}(k)^2 Z + \mathcal{N}(k)^2 N$ and for

$$\sum_{i>k}^{Z} \sum V(i,k) ,$$

in (1-*bis*) the expression

$$\mathcal{P}(k)^2 \sum_{i>k}^{Z}\sum \cos \boldsymbol{k}(\boldsymbol{R}_i - \boldsymbol{R}_k) + \mathcal{N}(k)^2 \sum_{r>1}^{N}\sum \cos \boldsymbol{k}\cdot(\boldsymbol{R}_r - \boldsymbol{R}_s) + \\ + \mathcal{P}(k)\mathcal{N}(k) \sum_{i=1}^{Z}\sum_{r=1}^{N} \cos \boldsymbol{k}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_r) .$$

As it is seen, a part from the presence of the factors $\mathcal{P}(k)$ and $\mathcal{N}(k)$ there are this time terms which correspond to the interaction of the neutrons among themselves and terms corresponding to the interaction between neutrons and protons. Calculating the mean value relative to all these interactions for the nucleus in its ground state and again using the rule of making the exchange contributions correspond to the incoherent scattering and the direct contributions correspond to the coherent scattering, the following expression is obtained for the incoherent form factor $\mathcal{I}$

$$Z^{-2}\left[\mathcal{P}(k)^2 Z + \mathcal{N}(k)^2 N - \mathcal{P}(k)^2 \sum_{i=1}^{Z}\sum_{k=1}^{Z} \left({}^{(p)}\psi_k^*(1)\,{}^{(p)}\psi_i^*(2)\,|\,V(1,2)\,|\,{}^{(p)}\psi_i(1)\,{}^{(p)}\psi_k(2)\right) - \right. \\ \left. - \mathcal{N}(k)^2 \sum_{i=1}^{N}\sum_{k=1}^{N} \left({}^{(n)}\psi_k^*(1)\,{}^{(n)}\psi_i^*(2)\,|\,V(1,2)\,|\,{}^{(n)}\psi_i(1)\,{}^{(n)}\psi_k(2)\right)\right],$$

where ${}^{(p)}\psi_i$ are the wave functions of the single protons, and ${}^{(n)}\psi_i$ are the wave functions of the single neutrons; this expression can be put in the form

$$Z^{-2}\left\{\mathcal{P}(k)^2\left[Z - \sum_{i=1}^{Z}\sum_{k=1}^{Z} |\,({}^{(p)}\psi_k^*\,|\,\exp[i\boldsymbol{k}\cdot\boldsymbol{R}]\,|\,{}^{(p)}\psi_i)\,|^2\right] + \right. \\ \left. + \mathcal{N}(k)^2\left[N - \sum_{i=1}^{N}\sum_{k=1}^{N} |\,({}^{(n)}\psi_k^*\,|\,\exp[i\boldsymbol{k}\cdot\boldsymbol{R}]\,|\,{}^{(n)}\psi_i)\,|^2\right]\right\}.$$

For the coherent contribution the following expression is obtained:

$$2\mathcal{P}(k)\mathcal{N}(k) \sum_{i=1}^{Z}\sum_{k=1}^{N} \left({}^{(p)}\psi_i^*(1)\,{}^{(n)}\psi_k^*(2)\,|\,V(1,2)\,|\,{}^{(p)}\psi_i(1)\,{}^{(n)}\psi_k(2)\right) + \\ + \mathcal{P}(k)^2 \sum_{i=1}^{Z}\sum_{k=1}^{Z} \left({}^{(p)}\psi_i^*(1)\,{}^{(p)}\psi_k^*(2)\,|\,V(1,2)\,|\,{}^{(p)}\psi_i(1)\,{}^{(p)}\psi_k(2)\right) + \\ + \mathcal{N}(k)^2 \sum_{i=1}^{N}\sum_{k=1}^{N} \left({}^{(n)}\psi_i^*(1)\,{}^{(n)}\psi_k^*(2)\,|\,V(1,2)\,|\,{}^{(n)}\psi_i(1)\,{}^{(n)}\psi_k(2)\right).$$

An expression of the type

$$\sum_{i=1}^{Z}\sum_{k=1}^{Z} \left({}^{(p)}\psi_i^*(1)\,{}^{(p)}\psi_k^*(2)\,|\,V(1,2)\,|\,{}^{(p)}\psi_i(1)\,{}^{(p)}\psi_k(2)\right)$$

gives

$$\left| \int \mathrm{d}\boldsymbol{R} \lambda^{(p)} \exp\left[i\boldsymbol{k}\cdot\boldsymbol{R}\right] \right|^2 = |[p]|^2,$$

provided that one puts

$$\sum_{i=1}^{Z} |{}^{(p)}\psi_i|^2 = \lambda^{(p)}$$

and it is clear what is meant by $[p]$; similarly for the neutrons, substituting the index n for p; the expression

$$\sum_{i=1}^{Z}\sum_{k=1}^{N} ({}^{(p)}\psi_i^*(1)\,{}^{(n)}\psi_k^*(2)\,|\,V(1,2)\,|\,{}^{(p)}\psi_i(1)\,{}^{(n)}\psi_k(2))$$

gives in the same manner $\frac{1}{2}[p][n]^* + \frac{1}{2}[p]^*[n]$. Thus the complete expression for the coherent form factor $\mathcal{C}$ is given by

$$Z^{-2}\{\mathcal{P}(k)\mathcal{T}(k)[p][n]^* + \mathcal{P}(k)\mathcal{T}(k)[p]^*[n] + \mathcal{P}(k)^2|[n]|^2 + \mathcal{T}(k)^2|[p]|^2\} =$$
$$= Z^{-2}|\mathcal{P}(k)[p] + \mathcal{T}(k)[n]|^2 .$$

The incoherent contribution is considered first (see also Fig. 8 relative to carbon). It is clear that at all angles this contribution is less than the

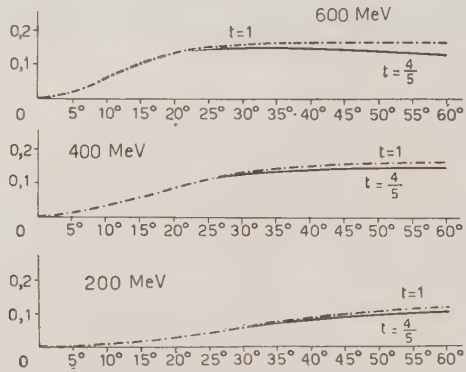

Fig. 8.

expression $Z^{-1}[\mathcal{P}(k)^2 + \alpha\mathcal{T}(k)^2]$ with $\alpha = N/Z$. A simple discussion shows that for a given α the expression $\mathcal{P}(k)^2 + \alpha\mathcal{T}(k)^2$ is less than 1 at all angles provided that $t > (\alpha - 1)/(\alpha + 1)$ and the maximum value of the latter expression for $1 \leqslant \alpha \leqslant 1{,}5$ is 0.2 for $\alpha = 1{,}5$, a value which is too far from 1 for the present theory to be applicable. It can therefore be concluded that at all angles, even in nuclei with a strong isotopic excess, there is a diminution in the incoherent scattering, if the contribution of the virtual mesons is added to the cross-section, and if this addition is made following a weak-coupling

theory. Passing now to the consideration of the coherent part, suppose for simplicity that the nucleons are distributed in the nucleus with constant density. In this case the form factor of the coherent scattering $\mathcal{C}$ is given by $[\mathcal{P}(k) + \mathcal{T}(k)\alpha]^2\mathcal{C}_0$ where we write $\mathcal{C}_0$ for the form factor which one would have without considering the contribution of the virtual mesons. This form factor can already be regarded to be zero, because of the finite dimensions of the nucleus, for $kR_0 \cong 4{,}49$ from which the minimum value of $\mathcal{F}(k)$, $\mathcal{F}_{\min} \cong \simeq (0{,}45A^{1/3})$ arctg $(2{,}24A^{-1/3})$ can be obtained, from which, in its turn, it is easy to obtain the maximum value of $(\mathcal{P} + \mathcal{T}\alpha)^2$, which for a nucleus like lead is $\cong 1{,}01$ for $t = 4/5$, and increases almost up to 1,02 for $t = 1/2$. For a nucleus with less isotopic excess, such as iron, only 1.006 would be obtained for $t = 4/5$. It can be said in conclusion that although the result is in the right direction qualitatively, that is, that a large excess of neutrons can lead to an increase in the cross-section at large angles, which seems to be suggested by present experimental results, quantitatively one is dealing with contributions presumably very small, certainly smaller than other errors of evaluation, for example errors due to the use of Born's approximation.

I would like to thank Prof. Edoardo Amaldi, who encouraged me to investigate these questions, and Prof. Bruno Ferretti, for their constant interest and many stimulating discussions.

I also thank Prof. S. Sciuti and Dr. B. Touschek for useful discussions and Prof. Mauro Picone, the Director, and Dr. W. Gross of the Istituto Nazionale per le Applicazioni del Calcolo for some difficult numerical computations concerning the integral (8).

RIASSUNTO

In relazione a recenti esperienze di scattering, che hanno condotto a sezioni d'urto inaspettatamente grandi, si esamina se una più accurata valutazione dei contributi elettromagnetici alla sezione d'urto può diminuire questa discrepanza. Viene in particolare considerato il contributo incoerente e discussa la dipendenza dal modello. Per calcolare il contributo incoerente si fa uso della approssimazione di completezza: si discute la validità di tale approssimazione ed in alcuni casi si confrontano i risultati così ottenuti con quelli ottenuti senza fare uso della approssimazione di completezza. Si può concludere che il contributo incoerente non spiega le sezioni d'urto trovate.

On a Theorem in Non-Local Field Theories (*).

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(ricevuto il 14 Settembre 1953)

Summary. — The Peierls formalism, which defines the Poisson brackets in a covariant way, is applied to a non-local field theory. It is shown that the P.B.'s of the ingoing and outgoing fields, defined with the Yang-Feldman method, are equal to those of the free fields.

1. – Introduction.

The first solution to the problem of the quantization of fields with non-local interaction has been given by Bloch (1) using an extension of the method developed by Yang and Feldman and by Källén (2) for local fields. The extension consists in postulating for the asimptotic parts of the fields at $t = -\infty$ and $t = +\infty$ (ingoing and outgoing fields respectively) the commutation relations of the free fields. In the usual local theories the quantization of the field functions is made directly by giving the commutation rules at all points on an arbitrary space-like surface. This is possible because the solution of the field equations is unique for any arbitrary initial conditions on such a space-like surface. For non-local theories the initial values problem, that is the determination of which field quantities can be arbitrarily given at a given time so that the past and future development of the system be uniquely determined, as only recently been investigated.

Pauli (3) has shown that non-local fields also have an hamiltonian structure, in the sense that if one restricts oneself to the « normal class » of form factors,

(*) This work was carried out during a stay at the E.T.H. in Zürich.

(1) C. Bloch: *Dan. Mat. Fys. Medd.*, **27**, n. 8 (1952).

(2) C. N. Yang and D. Feldman: *Phys. Rev.*, **76**, 972 (1952); G. Källén: *Ark. f. Fys.*, **2**, 371 (1951).

(3) W. Pauli: *Nuovo Cimento*, **10**, 648 (1953).

i.e. to those for which the initial value problem has a unique answer similarly as for free fields, it is possible to prove (at least for classical fields) the existence of canonical fields variables.

In a paper which appeared at the same time CHRÉTIEN and PEIERLS (4) have shown that form factors with the properties of this normal class really exist. A program of canonical quantization then becomes possible at least in principle.

The actual construction of canonical quantities, however, is closely connected with the initial value problem, the solution of which is only possible by power series.

For quantized fields a general demonstration of the existence of canonical variables has not so far been given, although for a particular example canonical variables have been explicitly constructed to the first approximation in the coupling constant.

The complexity of the canonical structure of these fields indicates that a method of approach, in which one makes direct use of the Lagrange function only, would be very useful.

PEIERSLS (5) has developed a formalism with which it is possible to give a definition of the Poisson brackets starting from the action integral and without any reference to the canonical variables, and which gives the possibility of introducing the commutation and anticommutation rules in a covariant way. But even, if for classical fields the formalism is established in a general manner for hamiltonian and non-hamiltonian theories, it is not the same for the quantum case. With theories for which the canonical method may be used, one can show that it is possible to obtain the commutation rules for the field operators in all cases of practical interest, while in the case of functions of the field variables this is possible for a restricted class of Lagrange functions only. In this last case as well for non-hamiltonian theories one tests the consistency of the method by inspection.

One can see that, with a lagrangian density of the type proposed by KRISTENSEN and MØLLER (6), the method is applicable only by treating the field variables as commuting quantities. Ignoring in this paper the problem of the possible modification of the interaction in this lagrangian density, it seems to us worthwhile to give here the result of a (classical) calculation done in the frame of the Kristensen-Møller theory in order to obtain an indication of the characteristic properties of the Peierls formalism in connection with non-local theories.

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(4) M. C. CHRÉTIEN and R. E. PEIERLS: *Nuovo Cimento*, **10**, 668 (1953).
(5) R. E. PEIERLS: *Proc. Roy. Soc.*, A **214**, 143 (1952).
(6) P. KRISTENSEN and C. MØLLER: *Dan. Mat. Fys. Medd.*, **27**, n. 7 (1952).

2. – Bloch's Theorem.

To describe a system of nucleons interacting with scalar mesons we shall assume that the field equations can be derived from the following action principle:

$$(1)\qquad \delta\Big\{\int \mathrm{d}x'(-\tfrac{1}{2}\,\overline{\psi}\gamma_\mu\partial_\mu\psi + \partial_\mu\overline{\psi}\gamma_\mu\psi - M\overline{\psi}\psi) - \int \mathrm{d}x''(\tfrac{1}{2}\partial_\mu u\cdot\partial_\mu u - m^2 u^2) - g\iiint \mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x'''F(x', x'', x''')\overline{\psi}(x')u(x'')\psi(x''')\Big\} = 0\,,$$

where F, the form factor, is a function depending on three different points of space-time and each $\mathrm{d}x$ stands for $\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4$. The interaction term vanishes for $t\to\pm\infty$. The field equations are then:

$$(2)\qquad \begin{cases} (\gamma_\mu\partial_\mu + M)\psi(x') + g\iint dx''\,\mathrm{d}x'''\,F(x', x'', x''')u(x'')\psi(x''') = 0\,, \\ \overline{\psi}(x''')(-\gamma_\mu\partial_\mu + M) + g\iint dx'\,\mathrm{d}x''\,F(x', x'', x''')\overline{\psi}(x')u(x'') = 0\,, \\ (\Box - m^2)\,u(x'') - g\iint dx'\,dx'''\,F(x', x'', x''')\overline{\psi}(x')\psi(x''') = 0\,. \end{cases}$$

Introducing the retarded and advanced Green functions it is possible to replace the preceding system of differential equations by one of the two following systems of integral equations:

$$(3)\qquad \begin{cases} \psi(x) = \overset{\text{in}}{\psi}{}^{\text{out}}(x) + g\iiint dx'\,\mathrm{d}x''\,\mathrm{d}x'''\,F(x', x'', x''')\,\overset{\text{ret}}{S}{}^{\text{adv}}(x - x')u(x'')\psi(x''')\,, \\ \psi(x) = \overline{\overset{\text{in}}{\psi}}{}^{\text{out}}(x) + g\iiint dx'\,dx''\,dx'''\,F(x', x'', x''')\overline{\psi}(x')\,u(x'')\,\overset{\text{adv}}{S}{}^{\text{ret}}(x''' - x)\,, \\ u(x) = \overset{\text{in}}{u}{}^{\text{out}}(x) - g\iiint dx'\,\mathrm{d}x''\,\mathrm{d}x'''\,F(x', x'', x''')\overline{\psi}(x')\,\overset{\text{ret}}{\Delta}{}^{\text{adv}}(x - x'')\psi(x''')\,. \end{cases}$$

It is clear that the inhomogeneous terms (ingoing and outgoing fields) satisfy the free fields equations. From the properties of the Green functions it also follows that they represent asimptotically the true fields at $t = -\infty$ and $t = +\infty$. The equations can be formally solved by iteration.

If thus for example the ingoing fields are given, one can express the outgoing fields as functions of the ingoing ones. The quantization method used

by BLOCH is based on the assumption that the same commutation rules hold for the ingoing fields as for the free ones. It is then possible to derive from (3) the commutation rules for the true and for the outgoing fields. BLOCH has tested the consistency of the formalism by showing that the commutation rules for the outgoing fields are the same as for the ingoing ones.

In what follows we shall show that, using the Peierls formalism, it is possible to obtain without making a power series expansion the following result: The P.B.s of the ingoing and outgoing fields are the same and coincide with that of the free fields

$$\{u^0(y), u^0(x)\} = \Delta(y-x)\,, \qquad \{\psi^0(y), \overline{\psi}^0(x)\} = -S(y-x)\,,$$

$$\{u^0(y), \psi^0(x)\} = \{\psi^0(y), \psi^0(x)\} = \{\overline{\psi}^0(y), \overline{\psi}^0(x)\} = 0\,.$$

3. – The Peierls formalism.

If one modifies the lagrangian density by adding one of the terms:

(1) $$\lambda u(x'')\,\delta^{(4)}(x''-y)\,,$$

(2) $$\lambda\overline{\psi}(x')\,\delta^{(4)}(x'-y)\,,$$

(3) $$\lambda\psi(x''')\,\delta^{(4)}(x'''-y)\,,$$

where λ is an infinitesimal parameter, and if one writes the solution of the new field equations to the first approximation in λ in the form

$$A' = A + \lambda D_B A\,,$$

(where A is any field function and B is $u(y)$, $\overline{\psi}(y)$, $\psi(y)$ in the three different cases), one obtains systems of the following type (case 1):

$$(\Box - m^2) D_{u(y)}u(x) -$$
$$- g\iint \mathrm{d}x'\,\mathrm{d}x'''\,F(x', x, x''')\left[D_{u(y)}\overline{\psi}(x')\cdot\psi(x''') + \overline{\psi}(x')\cdot D_{u(y)}\psi(x''')\right] = -\delta^{(4)}(x-y)\,,$$

$$(\gamma_\mu\partial_\mu + M) D_{u(y)}\psi(x) +$$
$$+ g\iint \mathrm{d}x''\,\mathrm{d}x'''\,F(x, x'', x''')\left[D_{u(y)}u(x'')\cdot\psi(x''') + u(x'')\cdot D_{u(y)}\psi(x''')\right] = 0\,,$$

$$D_{u(y)}\overline{\psi}(x)(-\gamma_\mu\partial_\mu + M) +$$
$$+ g\iint \mathrm{d}x'\,\mathrm{d}x''\,F(x', x'', x)\left[D_{u(y)}\overline{\psi}(x')\cdot u(x'') + \overline{\psi}(x')\cdot D_{u(y)}u(x'')\right] = 0.$$

Requiring that $D_B A \to 0$ for $t \to -\infty$, the system has a unique solution called the retarded solution. The same condition for the time $t = +\infty$ gives the advanced solution represented by the symbol $\mathcal{A}_B A$.

For the case 1 these solutions are:

$$
\left\{
\begin{aligned}
& D_{u(\psi)} u(x) = \Delta^{\mathrm{ret}}(x - y) - \\
& \quad - g \iiint \mathrm{d}x' \, \mathrm{d}x'' \, \mathrm{d}x''' \, F(x', x'', x''') \, \Delta^{\mathrm{ret}}(x - x'') \left[D_{u(\psi)} \overline{\psi}(x') \cdot \psi(x''') + \right. \\
& \quad \left. + \overline{\psi}(x') \cdot D_{u(\psi)} \psi(x''') \right], \\
& D_{u(\psi)} \psi(x) = \\
& \quad = g \iiint \mathrm{d}x' \, \mathrm{d}x'' \, \mathrm{d}x''' \, F(x', x'', x''') S^{\mathrm{ret}}(x - x') \left[D_{u(\psi)} u(x'') \cdot \psi(x''') + \right. \\
& \quad \left. + u(x'') \cdot D_{u(\psi)} \psi(x''') \right], \\
& D_{u(\psi)} \overline{\psi}(x) = \\
& \quad = g \iiint \mathrm{d}x' \, \mathrm{d}x'' \, \mathrm{d}x''' \, F(x', x'', x''') \left[D_{u(\psi)} \overline{\psi}(x') u \cdot (x'') + \right. \\
& \quad \left. + \overline{\psi}(x') D_{u(\psi)} u(x'') \right] S^{\mathrm{adv}}(x''' - x),
\end{aligned}
\right. \tag{4}
$$

and, with simplified notations:

$$
\left\{
\begin{aligned}
& \mathcal{A}_{u(\psi)} u(x) = \Delta^{\mathrm{adv}}(x - y) - g \int \mathrm{d}(x) F \, \Delta^{\mathrm{adv}}(x - x'') \left[\mathcal{A} \overline{\psi} \cdot \psi + \overline{\psi} \cdot \mathcal{A} \psi \right]_{u(\psi)}, \\
& \mathcal{A}_{u(\psi)} \psi(x) = g \int \mathrm{d}(x) F S^{\mathrm{adv}}(x - x') \left[\mathcal{A} u \cdot \psi + u \cdot \mathcal{A} \psi \right]_{u(\psi)}, \\
& \mathcal{A}_{u(\psi)} \overline{\psi}(x) = g \int \mathrm{d}(x) F \left[\mathcal{A} \overline{\psi} \cdot u + \overline{\psi} \cdot \mathcal{A} u \right]_{u(\psi)} S^{\mathrm{ret}}(x''' - x).
\end{aligned}
\right. \tag{4'}
$$

The basic definition of the formalism is then:

$$
\{B, A\} = D_B A - \mathcal{A}_B A . \tag{5}
$$

One verifies that all the properties defining the P.B.s are satisfied, among others the following:

$$
\{AB, C\} = \{A, C\} B + A \{B, C\} . \tag{6}
$$

One can also show that the reciprocity relationship holds.

$$
D_B A = \mathcal{A}_A B . \tag{7}
$$

It is now easy, using (3), (4) and (4′), to obtain the following relations

$$D_{u(y)}u^{\mathrm{in}}(x) = \Delta^{\mathrm{ret}}(x-y)\,, \qquad D_{u(y)}\psi^{\mathrm{in}}(x) = D_{u(y)}\bar{\psi}^{\mathrm{in}}(x) = 0\,,$$

$$\mathcal{D}_{u(y)}u^{\mathrm{in}}(x) = \Delta^{\mathrm{adv}}(x-y) - g\int \mathrm{d}(x) F \Delta(x-x'')\,[\mathcal{D}\bar{\psi}\cdot\psi + \bar{\psi}\cdot\mathcal{D}\psi]_{u(y)}\,,$$

$$\mathcal{D}_{u(y)}\psi^{\mathrm{in}}(x) = g\int \mathrm{d}(x) F S(x-x')\,[\mathcal{D}u\cdot\psi + u\cdot\mathcal{D}\psi]_{u(y)}\,,$$

$$\mathcal{D}_{u(y)}\bar{\psi}^{\mathrm{in}}(x) = -g\int \mathrm{d}(x) F[\mathcal{D}\bar{\psi}\cdot u + \bar{\psi}\cdot\mathcal{D}u]_{u(y)} S(x'''-x).$$

To obtain the corresponding expressions for the case of the outgoing fields one has just to interchange D with $\mathcal{D}$ and the retarded with the advanced functions. It will now no longer be necessary to speak about outgoing fields.

Using (5), one gets after some calculation the following P.B.s:

$$(8)\quad \begin{cases} \{u^{\mathrm{in}}(y), u^{\mathrm{in}}(x)\} = \Delta(y-x) + g\int \mathrm{d}(x) F\,\Delta(x-x'')\,[\mathcal{D}\bar{\psi}\cdot\psi + \bar{\psi}\cdot\mathcal{D}\psi]_{u(y)} + \\ \qquad\qquad + g\int \mathrm{d}(x) F\,\Delta^{\mathrm{ret}}(y-x'')\,\{\bar{\psi}\psi,\, u^{\mathrm{in}}(x)\}\,, \\ \{u^{\mathrm{in}}(y), \psi^{\mathrm{in}}(x)\} = -g\int \mathrm{d}x F S(x-x')\,[\mathcal{D}u\cdot\psi + u\cdot\mathcal{D}\psi]_{u(y)} + \\ \qquad\qquad + g\int \mathrm{d}(x) F\,\Delta^{\mathrm{ret}}(y-x'')\,\{\bar{\psi}\psi,\, \psi^{\mathrm{in}}(x)\}\,, \\ \{u^{\mathrm{in}}(y), \bar{\psi}^{\mathrm{in}}(x)\} = g\int \mathrm{d}x F[\mathcal{D}\bar{\psi}\cdot u + \bar{\psi}\mathcal{D}u]_{u(y)} S(x'''-x) + \\ \qquad\qquad + g\int \mathrm{d}(x) F\,\Delta^{\mathrm{ret}}(y-x'')\,\{\bar{\psi}\psi,\, \bar{\psi}^{\mathrm{in}}(x)\}\,. \end{cases}$$

Starting from the action principle, modified by adding $\lambda\bar{\psi}$ or $\lambda\psi$, it is possible to obtain in the same way four other groups of relations:

$$(9)\quad \begin{cases} D_{\bar{\psi}(y)}\psi(x) = -S^{\mathrm{ret}}(x-y) + g\int d(x) F S^{\mathrm{ret}}(x-x')\,[Du\cdot\psi + u\cdot D\psi]_{\bar{\psi}(y)}\,, \\ D_{\bar{\psi}(y)}\bar{\psi}(x) = g\int \mathrm{d}(x) F[D\bar{\psi}\cdot u + \bar{\psi}\cdot Du]_{\bar{\psi}(y)} S^{\mathrm{adv}}(x'''-x), \\ D_{\bar{\psi}(y)}u(x) = -g\int \mathrm{d}(x) F\Delta^{\mathrm{ret}}(x-x'')\,[D\bar{\psi}\cdot\psi + \bar{\psi}\cdot D\psi]_{\bar{\psi}(y)}\,, \\ \{\bar{\psi}(y), \psi^{\mathrm{in}}(x)\} = S(x-y) - g\int \mathrm{d}(x) F S(x-x')\,[\mathcal{D}u\cdot\psi + u\cdot\mathcal{D}\psi]_{\bar{\psi}(y}\,, \\ \{\bar{\psi}(y), \bar{\psi}^{\mathrm{in}}(x)\} = g\int \mathrm{d}(x) F[\mathcal{D}\bar{\psi}\cdot u + \bar{\psi}\cdot\mathcal{D}u]_{\bar{\psi}(y)} S(x'''-x)\,, \\ \{\bar{\psi}(y), u^{\mathrm{in}}(x)\} = g\int \mathrm{d}(x) F\Delta(x-x'')\,[\mathcal{D}\bar{\psi}\cdot\psi + \bar{\psi}\cdot\mathcal{D}\psi]_{\bar{\psi}(y)}\,, \end{cases}$$

and

$$
(10)\quad
\begin{cases}
D_{\psi(y)}\overline{\psi}(x) = -\,S^{\mathrm{adv}}(y-x) + g\int \mathrm{d}(x)\,F[D\overline{\psi}\cdot u + \overline{\psi}\cdot Du]_{\psi(y)}S^{\mathrm{adv}}(x'''-x)\,,\\
D_{\psi(y)}\psi(x) = g\int \mathrm{d}(x)\,F S^{\mathrm{ret}}(x-x')\,[Du\cdot\psi + u\cdot D\mu\psi]_{\psi(y)}\,,\\
D_{\psi(y)}u(x) = -\,g\int \mathrm{d}(x)\,F\Delta^{\mathrm{ret}}(x-x'')\,[D\overline{\psi}\cdot\psi + \overline{\psi}\cdot D\psi]_{\psi(y)}\,,\\
\{\psi(y),\,\overline{\psi}^{\mathrm{in}}(x)\} = -\,S(y-x) + g\int \mathrm{d}(x)\,F[\mathcal{A}\overline{\psi}\cdot u + \overline{\psi}\cdot\mathcal{A}u]_{\psi(y)}S(x'''-x),\\
\{\psi(y),\,\psi^{\mathrm{in}}(x)\} = -\,g\int \mathrm{d}(x)\,F S(x-x')\,[\mathcal{A}u\cdot\psi + u\cdot\mathcal{A}\psi]_{\psi(y)}\,,\\
\{\psi(y),\,u^{\mathrm{in}}(x)\} = g\int \mathrm{d}(x)\,F\Delta(x-x'')\,[\mathcal{A}\overline{\psi}\cdot\psi + \overline{\psi}\cdot\mathcal{A}\psi]_{\psi(y)}\,.
\end{cases}
$$

One can now calculate explicitly all the P.B.s for the ingoing fields. It will be sufficient to show that the result is the same as for the corresponding free fields in the case of a particular P.B. For instance one has

$$
(11)\qquad \{u^{\mathrm{in}}(y),\,u^{\mathrm{in}}(x)\} = \{u^{0}(y),\,u^{0}(u)\} = \Delta(y-x)\,.
$$

Thus rewriting the first equation of (8) and using (6) and (7), one has:

$$
\{u^{\mathrm{in}}(y),\,u^{\mathrm{in}}(x)\} = \Delta(y-x) + g\int \mathrm{d}(2)F(2)\Delta(x-2)\,[D_{\psi(2)}u(y)\cdot\psi(2) + \overline{\psi}(2)\cdot D_{\psi(2)}u(y)] +
$$

$$
+\, g\int \mathrm{d}(1)\,F(1)\,\Delta^{\mathrm{ret}}(y-1)\,[\overline{\psi}(1)\,\{\psi(1),\,u^{\mathrm{in}}(x)\} + \{\overline{\psi}(1),\,u^{\mathrm{in}}(x)\}\,\psi(1)]\,.
$$

Remembering the third equation of (9) and of (10), the second term becomes:

$$
g^2\iint \mathrm{d}(2)\mathrm{d}(1)\,F(2)F(1)\Delta(x-2)\,\Delta^{\mathrm{ret}}(y-1)\,\{[D\overline{\psi}(1)\cdot\psi(1) + \overline{\psi}(1)\cdot D\psi(1)]_{\overline{\psi}(2)}\cdot\psi(2) -
$$

$$
+\,\overline{\psi}(2)\,[D\overline{\psi}(1)\cdot\psi(1) + \overline{\psi}(1)\cdot D\psi(1)]_{\psi(2)}\}\,.
$$

From the last equation of (9) and of (10) and by again applying the reciprocity relationship, the third term becomes:

$$
g^2\iint \mathrm{d}(1)\mathrm{d}(2)F(1)F(2)\Delta(x-2)\,\Delta^{\mathrm{ret}}(y-1)\,\{\overline{\psi}(1)\,[D_{\psi(2)}\psi(1)\cdot\psi(2) + \overline{\psi}(2)\cdot D_{\psi(2)}\psi(1)] +
$$

$$
+\,[D_{\overline{\psi}(2)}\overline{\psi}(1)\cdot\psi(2) + \overline{\psi}(2)\cdot D_{\psi(2)}\overline{\psi}(1)]\psi(1)\}\,.
$$

Because ψ and ψ are supposed to be commuting quantities (11) then follows immediately.

The author is glad to acknowledge his indebtness to Prof. W. PAULI for stimulating his interest in this argument.

Thanks are due to the Collegio Ghislieri of Pavia for financial support.

RIASSUNTO

Viene applicato a una teoria di campi a interazione non locale il formalismo di Peierls, che definisce le parentesi di Poisson in modo covariante, per mostrare che queste per i campi incidenti ed emergenti (definiti secondo il metodo di Yang-Feldman) sono uguali a quelle dei campi liberi.

Action de la basse température sur les émulsions nucléaires.

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(ricevuto il 5 Ottobre 1953)

Resumé. — On étudie le comportement des émulsions nucléaires aux basses températures (— 85° et — 180°). Il est possible de développer et d'observer des plaques qui ont été maintenues pendant plusieurs semaines à ces basses températures à condition de couler l'émulsion sur un support de celluloïd. Avec un développement poussé aux maximum les C_2 ne laissent apparaître aucune trace de particules α à — 180° et des traces d'α discontinues à — 85°. Dans les G_5, les traces de particules α sont enregistrées et se présentent en grains séparés à — 180° et sous forme de traces continues à — 85°. Les électrons de 0,5 MeV environ sont enregistrés à — 85°. La technique de l'exposition à basse température peut servir a discriminer les traces, à diminuer la diffusion des gaz tels que le Radon et enfin, à ralentir les actions chimiques des échantillons mis en contact avec le plaques nucléaires.

Introduction.

Nous avons étudié le comportement des émulsions photographiques nucléaires Ilford C_2 et G_5 aux températures de — 85° et — 180°.

L'emploi des basses températures a déjà été suggéré comme obturateurs thermiques [1] et comme méthode de discrimination [2].

Dans cette étude, notre but était d'une part, de trouver des conditions évitant l'effet chimique de certains métaux et composés organiques sur l'émulsion et, d'autre part, de doser le Ra par la méthode photographique. La diffi-

[1] M. Cosyns, C. C. Dilworth et G. P. S. Occhialini: *Bulletin du Centre de Phys. Nucl. de l'U.L.B.*, note n. 6 (1949).

[2] C. C. Dilworth, G. P. S. Occhialini et L. Vermaesen: *Bulletin du Centre de Phys. Nucl. de l'U.L.B.*, note n. 13*a* (1950).

culté de ce dosage provient de la perte de Ra par diffusion [3-4]; la basse température a permis d'éviter cette diffusion (note en cours).

Résistance mécanique.

Les plaques d'une centaine de μ d'épaisseur subissent lors d'un refroidissement à très basse température une tension entre le verre et l'émulsion qui fait craqueler et décoller cette dernière et la rend ainsi inobservable (photos).

Après différents essais, nous avons réussi à éviter ce décollement en coulant l'émulsion « in gel form » sur un support de celluloïd d'environ 1 mm d'épaisseur au lieu du support de verre habituel. Pour l'exposition à basse température, les plaques ainsi obtenues sont introduites dans un tube de verre qui plonge dans le mélange réfrigérant. Celui-ci est constitué par de l'air liquide pour obtenir — 180° et par un mélange d'acétone et de glace carbonique pour — 85°. La température est mesurée à l'aide d'un thermocouple.

Action de la basse température sur l'image latente formée à temperature ordinaire.

Des plaques exposées à une source d'α à temperature ordinaire et ensuite maintenues 24 heures à — 180° ne montrent, après développement, aucune différence avec les plaques témoins conservées à température ordinaire.

L'image latente formée à température ordinaire ne subit donc aucun effacement visibile à basse température, comme on pouvait s'y attendre.

Action de la basse température sur l'enregistrement des traces.

L'aspect des traces formées à basse température dépend de la nature de l'émulsion et du développement.

Dans ces expériences, nous avons utilisé des plaques C_2 et G_5 d'une centaine de μ d'épaisseur. Elles ont toutes été développées dans les mêmes conditions avec un révélateur à l'amidol par la méthode à 2 températures [2] (*). Les α sont obtenus par un dépôt de gouttes d'une solution de $2 \cdot 10^{-3}$ g Ra/cm^3

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[3] SCHNEIDER et MATITSCH: *Sitzungsber. der oesterr. Akad. d. Wiss. Mathem. naturw. Klasse*, Abt. II*a*, **161**, 4. bis 6. H. (1952).

[4] L. VIGNERON, R. CHASTEL, J. GENIN: *Comptes Rendus*, **236**, 2053 (1953).

(*) Imprégnation du révélateur à 4° pendant 90 min. Stade chaud à sec: 26° pendant 60 min. Le révélateur contenait 18 g de sulfite de Na, 5 g d'amidol et 2 g de KBr par litre d'eau.

à pH neutre et d'un volume d'environ $2 \cdot 10^{-3}$ cm³. Après un séchage rapide d'une heure, les plaques ont été exposées à basse température pendant un temps variant de 1 jour à plusieurs semaines.

Dans une deuxième expérience, nous avons imprégné des plaques G_5 au SO_4Li_2 et les avons exposées à une source de neutrons constituée par 0,5 g de Ra-Be entouré de 10 cm de Pb et d'un mètre d'H_2O. Les thermos contenant les plaques et les mélanges réfrigérants étaient immergés dans l'eau à 10 cm de la source. L'exposition a duré 4 heures. A chaque essai, on a exposé une plaque témoin à température ordinaire. Les résultats ont été groupés dans le tableau qui suit (voir photos).

Température	C_2	G_5
18°	Traces α épaisses voile de grains normal	Ra traces α épaisses Li très fort voile de grains, traces α, H^3 continues
— 85°	Trajectoires α visibles en grains séparés, voile de grains normal	Ra α continus; β de 0,5 MeV visibles; voile de grains normal Li fort voile de grains; bonne discrimination α, H^3; α continus, H^3 en grains séparés
— 180°	Pas de traces visibles	Ra α sous forme de grains séparés, fort pourcentage de perte; β pas visibles. Li α et H^3 en grains séparés, forte proportion de désintégrations du Li perdue.

Ra représente les expériences effectuées avec les gouttes de Ra, Li se rapporte aux expériences $Li(n\,\alpha)H_1^3$.

Suppression du voile dû aux effets chimiques.

La conservation à basse température est un moyen efficace de diminuer le voile dû à certains agents chimiques. Il est bien connu par exemple que le contact de certains métaux réduit AgBr en Ag métallique. L'action de l'Al, métal très réducteur est particulièrement violente. Il est impossible d'observer une plaque ayant été maintenue en contact avec l'Al à température ordinaire. Nous avons conservé une émulsion C_2 en contact avec une plaque d'Al polie pendant 9 jours à — 85°. Alors que la plaque témoin correspondante, con-

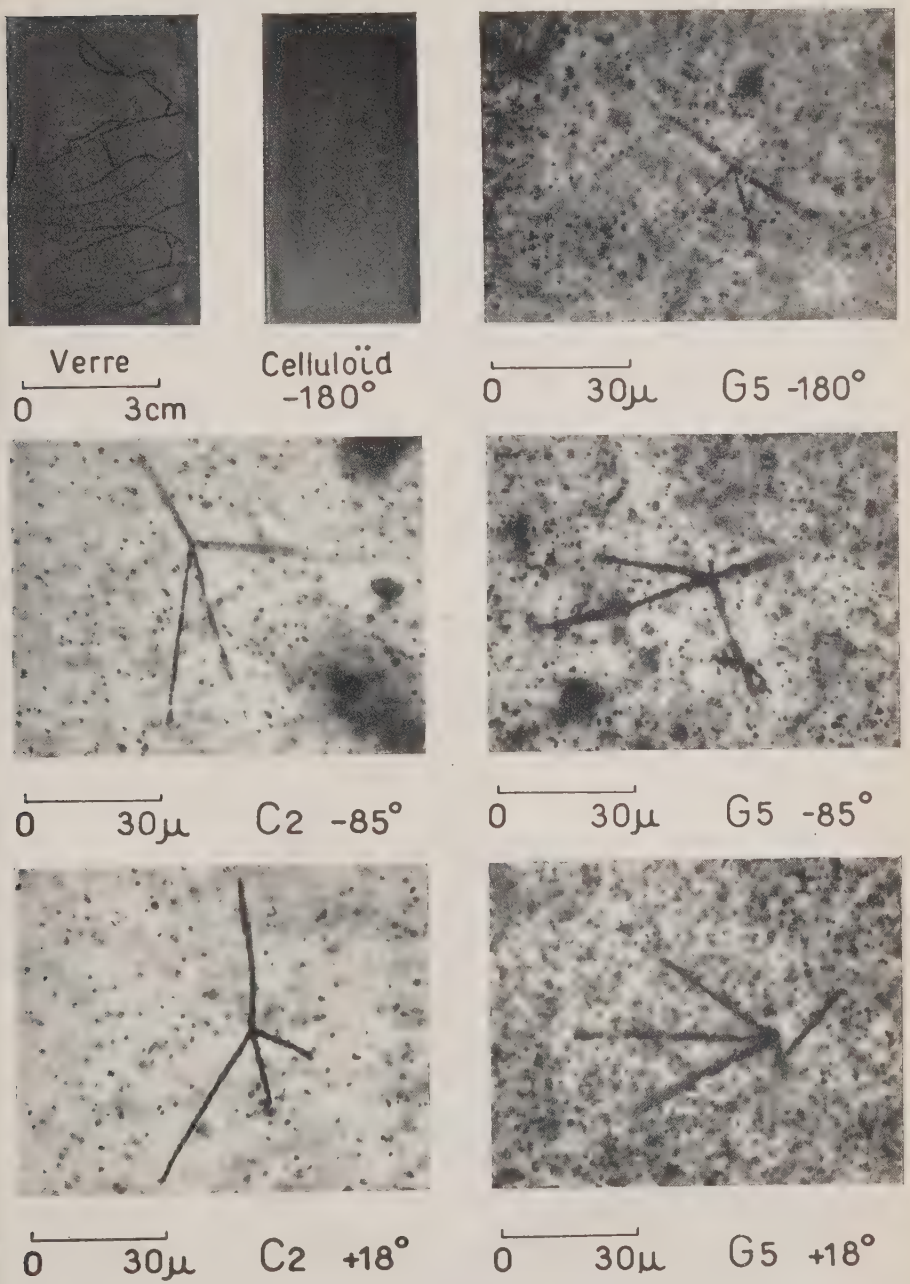
Verre
0 3cm
Celluloïd
-180°
0 30μ G5 -180°
0 30μ C2 -85°
0 30μ G5 -85°
0 30μ C2 +18°
0 30μ G5 +18°

Fig. 1.

servée à 4° est complètemente opaque, la plaque conservée à basse température est parfaitement observable et ne montre aucun voile anormal .

Conclusions.

Il est possible de conserver en excellent état pendant plusieurs semaines des émulsions nucléaires de 50 à 200 μ coulées sur un support de celluloïd à la température de l'air liquide. A — 180°, les émulsions C_2 sont insensibles aux α quel que soit le développement, les G_5 enregistrent très faiblement les α. A — 85°, les émulsions C_2 enregistrent les α sous forme de trajectoire en grains séparés, les émulsions G_5 avec un développement poussé enregistrent les α en trajectoires continues et les β d'environ 0,5 MeV.

A — 85°, le voile dû à l'action réductrice des métaux tels que l'Al est supprimée.

Ce travail a été effectué au Centre de Physique Nucléaire de l'Université Libre de Bruxelles sous la direction du Dr. E. Picciotto.

Je remercie le Professeur P. Baudoux de l'hospitalité qui m'a été accordée dans son laboratoire.

Je remercie tous les membres du Laboratoire et spécialement le Professeur Occhialini de leur collaboration et de leurs conseils.

Je suis très reconnaissante à l'Office des Relations Culturelles Franco-belges de la bourse qui m'a été octroyée pour mon séjour en Belgique.

RIASSUNTO (*)

Si studia il comportamento delle emulsioni nucleari alle basse temperature (— 85° e — 180°). È possibile sviluppare e osservare delle lastre mantenute durante più settimane a queste basse temperature a condizione di colare l'emulsione su un supporto di celluloide. Con sviluppo spinto al massimo le C_2 non mostrano alcuna traccia di particelle α a — 180° e tracce d'α discontinue a — 85°. Nelle G_5 le tracce di particelle α restano registrate e si presentano in granuli separati a — 180° e a — 85° sotto forma di tracce continue. Gli elettroni di circa 0,5 MeV risultano registrati a — 85°. La tecnica dell'esposizione a bassa temperatura può servire a discriminare le tracce, a diminuire la diffusione dei gas come il Radon e, finalmente, a rallentare le azioni chimiche dei campioni messi a contatto con le lastre nucleari.

(*) *Traduzione a cura della Redazione.*

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Dirac's Electrodynamics and Einstein's Unified Field Theory.

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(ricevuto il 1° Ottobre 1953)

The Einstein unified field theory [1] is based on the sixty-four equations

$$(1) \qquad g_{\underset{+-}{mn;k}} = \frac{\partial g_{mn}}{\partial x^k} - g_{ms}\Gamma^s_{kn} - g_{sn}\Gamma^s_{mk} = 0\,, \qquad (m, n, k = 1, 2, 3, 4)$$

which determine the components of the non-symmetric affine connection. It seems almost certain that these equations are the appropriate generalization of the symmetric set $g_{mn;k} = 0$ of the General Theory of Relativity, since, apart from the reasons of Hermitian symmetry discussed by EINSTEIN, HLAVATÝ [2] has shown that the $(+\,-)$ covariant differentiation is necessary if there are to exist solutions in the Γ field in which the symmetric and skew-symmetric parts of g_{mn} are independent.

Splitting (1) into symmetric and skew-symmetric parts and permuting the indices cyclically, we find the implicit solutions

$$(2) \qquad \Gamma^s_{\underline{mn}} = \left\{ \begin{matrix} s \\ mn \end{matrix} \right\} + g^{\underline{sk}}(F_{pn}\Gamma^p_{\underset{\vee}{mk}} + F_{mp}\Gamma^p_{\underset{\vee}{kn}})$$

and

$$(3) \qquad \Gamma^s_{\underset{\vee}{mn}} = -\frac{1}{2}\, g^{\underline{sk}} H_{kmn} + g^{\underline{sk}} F_{mn|k}\,,$$

where F_{mn} has been written for $g_{\underset{\vee}{mn}}$, and $H_{kmn} = \dfrac{\partial F_{km}}{\partial x^n} + \dfrac{\partial F_{mn}}{\partial x^k} + \dfrac{\partial F_{nk}}{\partial x^m}$. The vertical bar denotes covariant differentiation with respect to the symmetric connection $\Gamma^s_{\underline{mn}}$, and $\left\{ \begin{matrix} s \\ mn \end{matrix} \right\}$ is the Christoffel bracket formed from the symmetric $g_{\underline{mn}}$. All indices are raised and lowered by the $g_{\underline{mn}}$.

(1) A. EINSTEIN: *The Meaning of Relativity* (London, 1950), Appendix 2.
(2) V. HLAVATÝ: *J. Rational Mech. Anal.*, **2**, 1 (1953).

We now modify the Einstein theory by requiring that the six-vector F_{mn} (which is associated with the six-vector of the electromagnetic field) shall be derivable from a four-potential A_m (say). A necessary condition for this requirement is expressed by

$$H_{kmn} = 0\,, \tag{4}$$

which is one set of Maxwell's equations.

Apart from equations (1) which determine the Γ field, the additional restriction $\Gamma_m\ (= \underset{\vee}{\Gamma^s_{ms}}) = 0$ is imposed in the Einstein theory. In place of this condition, we now impose the relation

$$\Gamma_m\Gamma^m = \text{constant}\,, \tag{5}$$

and identify Γ_m with the four-potential A_m. Contracting (3) and raising indices, we find

$$g^{\underline{ym}}g^{\underline{sk}}F_{ys|k} = A^m\,, \tag{6}$$

which forms a second set of electromagnetic equations with a charge-current density expressed in terms of the four-potential.

Solving (2) and (3) and neglecting cubes and higher powers of F_{mn}, we find

$$\Gamma^s_{\underline{mn}} = \left\{ \begin{matrix} s \\ mn \end{matrix} \right\} + g^{\underline{sk}}g^{\underline{i\omega}}(F_{\underline{in}}F_{mk||\omega} + F_{mi}F_{kn||\omega}) \tag{7}$$

and

$$\underset{\vee}{\Gamma^s_{mn}} = g^{\underline{sk}}F_{mn||k}\,, \qquad (\text{since}\quad H_{kmn} = 0) \tag{8}$$

where $||$ denotes covariant differentiation with respect to the Christoffel bracket $\left\{ \begin{matrix} s \\ mn \end{matrix} \right\}$. To the same approximation, (6) becomes

$$F^{ms}{}_{||s} = A^m\,. \tag{9}$$

These equations are the Riemannian curved space (metric tensor $g_{\underline{mn}}$) generalization of the special relativistic electromagnetic equations recently proposed by Dirac [3] in his non-gauge invariant theory of electrons, whilst (5) corresponds to his gauge destroying condition $A_mA^m = \text{constant}$. The remaining field equations of the Einstein theory (which include the equations for the gravitational field) are expressed in terms of the Hermitian curvature tensor R_{mn} which may be calculated using (7) and (8).

It is the purpose of this note to show that by imposing different conditions in the Einstein unified theory different electromagnetic theories result. In particular we find that condition (5) allows the formulation of the generalization of the Dirac eletrrodynamics (equations (6)). In the weak field approximation, the theory is equivalent to the intuitive formulation of the Dirac theory in a Riemannian space, as shown by equations (9).

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[3] P. A. M. Dirac: *Proc. Roy. Soc.*, A **209**, 291 (1951).

IL NUOVO CIMENTO Vol. X, N. 11 1° Novembre 1953

Generalization of the Classical Field Formalism by Means of Functionals.

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(ricevuto il 2 Ottobre 1953)

Schwarz [1] showed that the ordinary concept of function can be generalized by means of the linear functionals. Thus the use of symbolic functions such as the Dirac δ function and its derivatives can be avoided. We have applied recently [2-4] both linear and non linear functionals to the quantum mechanics, in order to obtain a statistical generalization of the quantal formalism. In our treatment of the quantum mechanics by means of functionals, the wave function $\Psi(t; \boldsymbol{x})$ is replaced by a functional $\chi[t; \psi^*(\boldsymbol{x})]$, $\psi^*(\boldsymbol{x})$ being of the same nature as the complex conjugate of the wave function Ψ for a fixed value t_0 of the time. The linear functionals $\chi_1[t; \psi^*(\boldsymbol{x})]$

$$\chi_1[t; \psi^*] = \int \Psi(t; \boldsymbol{x})\psi^*(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{1}$$

describe the same pure states as the kernel functions $\Psi(t; \boldsymbol{x})$. The quadratic functionals $\chi_2[t; \psi^*]$

$$\chi_2[t; \psi^*] = \int \Psi_2(t; \boldsymbol{x}_1, \boldsymbol{x}_2)\psi^*(\boldsymbol{x}_1)\psi^*(\boldsymbol{x}_2)\,\mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{x}_2 \tag{2}$$

describe the same mixtures as the von Neumann density matrices $\langle \boldsymbol{x}_1 | R(t) | \boldsymbol{x}_2 \rangle$

$$\langle \boldsymbol{x}_1 | R(t) | \boldsymbol{x}_2 \rangle = \int \Psi_2(t; \boldsymbol{x}_1, \boldsymbol{x})\Psi_2^*(t; \boldsymbol{x}, \boldsymbol{x}_2)\,\mathrm{d}\boldsymbol{x}. \tag{3}$$

We have shown in reference 4 that all the von Neumann density matrices can be obtained from quadratic functionals $\chi_2[t; \psi^*]$ by the application of the rule (3). The χ-formalism offers some essentially new possibilities of statistical generalization of the quantum mechanics. We have discussed in reference 4 the important func-

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[1] L. Schwarz: *Théorie des distributions* (Paris, 1950, 1951).
[2] M. Schönberg: *Nuovo Cimento*, **10**, 350 (1953).
[3] M. Schönberg: *Nuovo Cimento*, **10**, 697 (1953).
[4] M. Schönberg: *A statistical generalization of the quantum mechanics I*, in *Nuovo Cimento* (in print).

tional $\Delta[\psi^*, \Psi(t; \boldsymbol{x})]$

$$\Delta[\psi^*, \Psi(t; \boldsymbol{x})] = \exp\left\{\int \Psi(t; \boldsymbol{x})\psi^*(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right\}, \tag{4}$$

which leads to an interesting generalization of the pure states of the ordinary quantal formalism. The relations between the χ-formalism and the second quantization of the Schrödinger equation were discussed in reference **4**. The equation of evolution of the functionals $\chi[t; \psi^*]$

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\chi[t; \psi^*] = \int \psi^*(\boldsymbol{x}) H \frac{\delta\chi}{\delta\psi^*(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}, \quad (H = \text{hamiltonian operator}), \tag{5}$$

has the same form as the Schrödinger equation of the quantized Ψ-field with Bose statistics, in the representation in which the creation operators are diagonal, the creation operators being the complex conjugates of the quantized wave function, as usual.

The Liouville equation of the classical wave field described by Ψ is

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[t; \psi^*, \psi] = \int \left\{\psi^*(\boldsymbol{x}) H \frac{\delta\mathcal{F}}{\delta\psi^*(\boldsymbol{x})} - \frac{\delta\mathcal{F}}{\delta\psi(\boldsymbol{x})} H\psi(\boldsymbol{x})\right\}\mathrm{d}\boldsymbol{x}\,. \tag{6}$$

In the particular case of a functional $\mathcal{F}[t; \psi^*]$, equation (6) goes over into (5). Thus the χ-formalism is closely related to the classical theory of the constants of motion of the Ψ-field. The equation (5) expresses simply that $\chi[t; \psi^*]$ is a constant of the motion of the classical Ψ-field.

The method used to generalize the ordinary formalism of the quantum mechanics can be extended to other classical field theories which admit a variational principle. Let us denote by $\Phi_\alpha(\boldsymbol{x})$ the variables which describe such a field, $\Pi_\alpha(\boldsymbol{x})$ the corresponding conjugate momenta and $\mathcal{H}[\Phi, \Pi]$ the hamiltonian of the field. The equation for the constants of motion of the Φ-field (Liouville equation) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[t; \Phi, \Pi] = \int \sum_\alpha \left\{\frac{\delta\mathcal{H}}{\delta\Phi_\alpha(\boldsymbol{x})}\frac{\delta\mathcal{F}}{\delta\Pi_\alpha(\boldsymbol{x})} - \frac{\delta\mathcal{H}}{\delta\Pi_\alpha(\boldsymbol{x})}\frac{\delta\mathcal{F}}{\delta\Phi_\alpha(\boldsymbol{x})}\right\}\mathrm{d}\boldsymbol{x}\,. \tag{7}$$

In the general case of non linear field equations it is not possible to obtain from equation (7), which corresponds to (6), an equation corresponding to (5). In the particular case of linear field equations, the hamiltonian $\mathcal{H}$ is quadratic or bilinear in the Φ_α and Π_α. This special structure of $\mathcal{H}$ allows us to get from (7) an equation corresponding to (5), at least in the cases ordinarily considered. Let us examine as an example the field described by a real scalar Φ satisfying the Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \Delta\Phi + K^2\Phi = 0\,. \tag{8}$$

The conjugate momentum of Φ is $(1/c^2)(\partial\Phi/\partial t)$ and the hamiltonian of the field is

$$\mathcal{H} = \frac{1}{2}\int \left\{\left(\frac{1}{c}\frac{\partial\Phi}{\partial t}\right)^2 + \left(\frac{\partial\Phi}{\partial\boldsymbol{x}}\right)^2 + K^2\Phi^2\right\}\mathrm{d}\boldsymbol{x}\,. \tag{9}$$

We shall introduce the new field variables Φ and Φ^* defined by the equations

$$\Phi = \frac{1}{\sqrt{2}}(\Phi + \Phi^*), \qquad \frac{1}{c}\frac{\partial \Phi}{\partial t} = \frac{1}{\sqrt{2}}\sqrt{K^2 - \Delta}\,(\Phi^* - \Phi). \tag{10}$$

The $\Phi^*(\boldsymbol{x})$ will be considered as the new field coordinates, the corresponding conjugate momenta being the $\Pi_{\Phi^*(\boldsymbol{x})} = -(i/c)\sqrt{K^2 - \Delta}\,\Phi(\boldsymbol{x})$. By taking into account that $\sqrt{K^2 - \Delta}$ is a hermitian operator, we get

$$\mathcal{H} = ic \int \Pi_{\Phi^*(\boldsymbol{x})}\sqrt{K^2 - \Delta}\,\Phi^*(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,. \tag{11}$$

The Liouville equation is now

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = ic \int \left\{ \frac{\delta \mathcal{F}}{\delta \Pi_{\Phi^*(\boldsymbol{x})}}\sqrt{K^2 - \Delta}\,\Pi_{\Phi^*(\boldsymbol{x})} - \frac{\delta \mathcal{F}}{\delta \Phi^*(\boldsymbol{x})}\sqrt{K^2 - \Delta}\,\Phi^*(\boldsymbol{x}) \right\} \mathrm{d}\boldsymbol{x}\,. \tag{12}$$

We get the analogue of (5) by taking the Liouville equation for constants of the motion which do not depend on the momenta

$$i\frac{\mathrm{d}}{\mathrm{d}t}\chi[t;\, \Phi^*(\boldsymbol{x})] = c \int \Phi^*(\boldsymbol{x})\sqrt{K^2 - \Delta}\;\frac{\delta \chi}{\delta \Phi^*(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}\,. \tag{13}$$

In the present case the χ-formalism is also related to the theory of the quantized Φ-field. The quantization of the Φ-field can be performed by replacing $\Phi^*(\boldsymbol{x})$ and $\Phi(\boldsymbol{x})$ by operators satisfying the commutation rules

$$\begin{cases} [\Phi_{\mathrm{op}}(\boldsymbol{x}),\, \Phi_{\mathrm{op}}(\boldsymbol{x}')] = [\Phi^*_{\mathrm{op}}(\boldsymbol{x}),\, \Phi^*_{\mathrm{op}}(\boldsymbol{x}')] = 0 \\ [\Phi^*_{\mathrm{op}}(\boldsymbol{x}),\, \sqrt{K^2 - \Delta}\,\Phi_{\mathrm{op}}(\boldsymbol{x}')] = -\hbar c\,\delta(\boldsymbol{x} - \boldsymbol{x}'). \end{cases} \tag{14}$$

In the representation in which the $\Phi^*_{\mathrm{op}}(\boldsymbol{x})$ are diagonal, we can take

$$\sqrt{K^2 - \Delta}\,\Phi_{\mathrm{op}}(\boldsymbol{x}) = \hbar c\,\frac{\delta}{\delta \Phi^*(\boldsymbol{x})}\,. \tag{15}$$

Hence the Schrödinger equation for the wave functional $\Omega[t;\, \Phi^*]$ becomes

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Omega[t;\, \Phi^*] = \mathcal{H}_{\mathrm{op}}\Omega[t;\, \Phi^*] = \frac{\hbar c}{2}\int \sqrt{K^2 - \Delta}\,\Phi^*(\boldsymbol{x})\,\frac{\delta \Omega}{\delta \Phi^*(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} + \tag{16}$$

$$+ \frac{\hbar c}{2}\int \frac{\delta}{\delta \Phi^*(\boldsymbol{x})}\left\{\sqrt{K^2 - \Delta}\,\Phi^*(\boldsymbol{x})\Omega\right\}\mathrm{d}\boldsymbol{x}\,,$$

$$i\frac{\mathrm{d}}{\mathrm{d}t}\Omega[t;\, \Phi^*] = c\int \Phi^*(\boldsymbol{x})\sqrt{K^2 - \Delta}\;\frac{\delta \Omega}{\delta \Phi^*(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} + \tag{17}$$

$$+ \frac{c}{2}\,\Omega\int \frac{\delta}{\delta \Phi^*(\boldsymbol{x})}\left\{\sqrt{K^2 - \Delta}\,\Phi^*(\boldsymbol{x})\right\}\mathrm{d}\boldsymbol{x}\,.$$

In the derivation of (16) we used the hamiltonian $\mathcal{H}_{\mathrm{op}}$ of the quantized Φ-field

$$\mathcal{H}_{\mathrm{op}} = \frac{1}{2}\int \{\sqrt{K^2 - \Delta}\,\Phi^*_{\mathrm{op}}(\boldsymbol{x})\sqrt{K^2 - \Delta}\,\Phi_{\mathrm{op}}(\boldsymbol{x}) + \sqrt{K^2 - \Delta}\,\Phi_{\mathrm{op}}(\boldsymbol{x})\sqrt{K^2 - \Delta}\,\Phi^*_{\mathrm{op}}(\boldsymbol{x})\}\,\mathrm{d}\boldsymbol{x} \tag{18}$$

It is remarkable that the Planck constant cancel out from the Schrödinger equation of the quantized Φ-field in the Φ^*-representation. The Schrödinger equation (17) *differs only from the equation of evolution* (13) *of the functional $\chi[t; \Phi^*]$ by the term arising from the infinite zero-point energy of the quantized Φ-field, i.e. the second term in the right hand side of* (17).

Let us introduce the Fourier integral expansion of $\Phi(\boldsymbol{x})$

$$\Phi(\boldsymbol{x}) = (2\pi)^{-3/2}\int a(\boldsymbol{k})\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{k}. \tag{19}$$

We have

$$\sqrt{K^2 - \Delta}\,\Phi(\boldsymbol{x}) = (2\pi)^{-3/2}\int k_0 a(\boldsymbol{k})\exp[i\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{k}\,, \qquad k_0 = \sqrt{k^2 + K^2} \tag{20}$$

$$\mathcal{H} = \int |a^*(\boldsymbol{k})|^2 k_0^2\,\mathrm{d}\boldsymbol{k}\,, \qquad \mathcal{H}_{\mathrm{op}} = \frac{1}{2}\int \{a^*_{\mathrm{op}}(\boldsymbol{k})a_{\mathrm{op}}(\boldsymbol{k}) + a_{\mathrm{op}}(\boldsymbol{k})a^*_{\mathrm{op}}(\boldsymbol{k})\}\,k_0^2\,\mathrm{d}\boldsymbol{k}\,. \tag{21}$$

The conjugate momentum of $a^*(\boldsymbol{k})$ is $-(ik_0/c)a(\boldsymbol{k})$. The constants of the motion may be regarded as functionals $\mathcal{F}[t; a^*, a]$ of $a^*(\boldsymbol{k})$ and $a(\boldsymbol{k})$. Hence the Liouville equation becomes

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = ic\int\left\{a(\boldsymbol{k})\frac{\delta\mathcal{F}}{\delta a(\boldsymbol{k})} - a^*(\boldsymbol{k})\frac{\delta\mathcal{F}}{\delta a^*(\boldsymbol{k})}\right\}k_0\,\mathrm{d}\boldsymbol{k}\,. \tag{22}$$

The constants of the motion which depend only on the function $a^*(\boldsymbol{k})$ satisfy the equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\chi[t; a^*(\boldsymbol{k})] = c\int a^*(\boldsymbol{k})\frac{\delta\chi}{\delta a^*(\boldsymbol{k})}k_0\,\mathrm{d}\boldsymbol{k}\,. \tag{23}$$

In the representation in which the $a^*_{\mathrm{op}}(\boldsymbol{k})$ are diagonal we can take $a_{\mathrm{op}}(\boldsymbol{k}) = (\hbar c/k_0)(\delta/\delta a^*(\boldsymbol{k}))$ and the Schrödinger equation of the quantized Φ-field becomes

$$i\frac{\mathrm{d}}{\mathrm{d}t}\Omega[t; a^*] = c\int a^*(\boldsymbol{k})\frac{\delta\Omega}{\delta a^*(\boldsymbol{k})}k_0\,\mathrm{d}\boldsymbol{k} + \frac{c}{2}\,\Omega\int\frac{\delta a^*(\boldsymbol{k})}{\delta a^*(\boldsymbol{k})}k_0\,\mathrm{d}\boldsymbol{k}\,. \tag{24}$$

The mixtures of states of the quantized Φ-field are described by density matrices whose matrix elements are functionals $\langle a'^*|\mathcal{R}(t)|a''\rangle$ of two functions $a'^*(\boldsymbol{k})$ and $a''(\boldsymbol{k})$. $\langle a'^*|\mathcal{R}|(t)|a''\rangle$ satisfies the von Neumann equation for the quantized

Φ-field

$$(25)\qquad i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle a'|\mathcal{R}(t)|a''\rangle =$$

$$= \mathcal{H}_{\mathrm{op}}[a'^*]\langle a'^*|\mathcal{R}(t)|a''\rangle - \{\mathcal{H}_{\mathrm{op}}[a''^*]\langle a'^*|\mathcal{R}(t)|a'\rangle\}^* =$$

$$= \hbar c\int \mathrm{d}\boldsymbol{k}\,k_0\left\{a'^*(\boldsymbol{k})\frac{\delta}{\delta a'^*(\boldsymbol{k})} - a''(\boldsymbol{k})\frac{\delta}{\delta a''(\boldsymbol{k})}\right\}\langle a'^*|\mathcal{R}(t)|a''\rangle\,.$$

The diagonal matrix elements $\langle a'^*|\mathcal{R}(t)|a'\rangle$ *satisfy the Liouville equation* (22) *of the classical* Φ*-field.*

The analogue of the functional $A[\psi^*, \Psi]$ defined by (4) in the case of the Φ-field is $A[a^*(\boldsymbol{k}), (k_0/\hbar c)A(\boldsymbol{k})]$ which describes the state of the quantized Φ-field corresponding to the eigenvalues $A(\boldsymbol{k})$ of the $a_{\mathrm{op}}(\boldsymbol{k})$. It is interesting to notice that the Planck constant which appears in $A[a^*, (k_0/\hbar c)A]$ is necessary to render dimensionless the exponent $(\hbar c)^{-1}\int A(\boldsymbol{k})a^*(\boldsymbol{k})k_0\,\mathrm{d}\boldsymbol{k}$. Without the introduction of a constant with the dimension of an action, we are restricted to the consideration of very special types of functionals in the χ-formalism of the Φ-field.

A more detailed treatment of the χ-formalism will appear later in this journal.

On a Non-Local Field Theory.

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(ricevuto il 23 Ottobre 1953)

In the present note (1) we want to suggest a new form of a non-local field theory, which starts from the S-matrix formalism deduced by means of a perturbation theory (Dirac-Feynman-Dyson) based on the choice of a hamiltonian and the use of the interaction representation.

From the requirements of complementarity follows that if one wants to describe the exact validity of conservation laws of momenta in a collision process, one is compelled to represent the initial and final states of particles as eigen-states of momenta (plane-waves) and therefore one is induced to adopt the S-matrix formalism. The description of propagation in space and time in this case is limited to the assumption of approximate validity of the laws of propagation of free particles outside the domain of interaction (in analogy to the methods of the R-matrix used by Wigner and Eisenbud to describe the nuclear reactions).

Let us consider the example of a collision of a neutral pion and a proton, and a local hamiltonian density:

$$H' = ig_1\psi^+\gamma_5\psi\Phi - i\frac{g_2}{m}\psi^+\gamma_5\gamma_\mu\partial_\mu\Phi$$

(here: $\hbar = 1$, $c = 1$, ψ, Φ are normalized representatives of the states of free particles). In the well known integrals appearing in the calculation of terms of the S-matrix we shall decompose the integrand in a sum of products of representatives of states with positive and negative energies and we shall make the calculations in the momentum space.

Let us examine a term of the S-matrix corresponding to a given Feynman diagram of n vertices and containing in the initial state one incident proton and one incident pion (two « ingoing » external lines) and a given number of nucleons and mesons created in final states (« outgoing » external lines). In the momentum space

(1) In recent years many researches were made on the non-local field theories. The present work is closely connected to the early attempt of the author; *Zeits., f. Phys.*, **88**, 92 (1934).

this term of the S-matrix will contain the absorption and the creation operators and spinors or polarization-vectors corresponding to the ingoing and outgoing lines. There will be also a set of operators for the vertices and of propagation factors for the internal lines.

In order to introduce in this formalism the rules of a non-local field theory we must begin with the definition of the relativistic cut-off operators. Let:

$$P_\mu = p'_\mu + p''_\mu \qquad \mu = 0, 1, 2, 3$$

be the total energy momentum vector of the incident particles (proton and pion) and let:

$$P_\mu P^\mu = P_0^2 - P_1^2 - P_2^2 - P_3^2 = m_0^2 > 0 .$$

We introduce the unit-vector velocity $u_\mu = (1/m_0)P_\mu$, and we associate to every normalized eigen-state of momentum p_μ of an external line an invariant cut-off operator $\mathcal{G}$ and an operator $\mathcal{G}^2$ to any internal line. According to the requirements of relativity, this operator $\mathcal{G}$ could depend on the following invariants:

$$I = p^\nu u_\nu , \qquad \mathcal{J} = p^\nu p_\nu , \qquad m_0^2 = P^\nu P_\nu$$

$$(p_\nu - P_\nu)(p^\nu - P^\nu) = \mathcal{J} + m_0^2 - 2m_0 I , \text{ etc.}$$

We shall discuss here a very simple example of an operator depending explicitly only on the invariants I and $\mathcal{J}$. Let: $a^2 = I^2 - \mathcal{J}$. This invariant becomes in the center of mass system (C-system, $u_0 = 1$, $u_1 = 0 = u_2 = u_3$):

$$a^2 = p_1^2 + p_2^2 + p_3^2 .$$

We define $\mathcal{G}$ as follows:

$$\mathcal{G} = \frac{3[\sin (al) - al \cos (al)]}{(al)^3} , \tag{1}$$

where l is a universal length. This choice corresponds in the C-system to the Fourier transform of the function $\delta(x_0)f(x_1x_2x_3)$, where f has a constant non zero value inside a spherical domain in the S_3 space of relative coordinates $(x_1x_2x_3)$, and vanishes outside this domain. (Here x_μ can be defined as (unobservable) differences between the particle's coordinates and the center of mass coordinates). For large values of a, $\mathcal{G}^2$ behaves as: $\cos^2 (al)/(al)^4$.

One has:

$$\lim_{a \to 0} \mathcal{G} = 1 .$$

The new prescriptions added to the rules of the calculation of the S-matrix can now be formulated as follows: in every term corresponding to the Feynman graph considered above we introduce the G-operator (1) in every external line and the $\mathcal{G}^2$ operator in every internal line. In all cases the 4-vector u_ν is defined by the total momentum and energy of the ingoing particles in the considered collision process, or also by the momentum and energy of the outgoing particles (because of the con-

servation laws). For the inverse process one obtains the same cut-off operators. In this way all the divergent integrals are made convergent. The hermitian property of the hamiltonian is conserved. In order to be sure to preserve the unitary character of the S-matrix, one should start from the hermitian matrix K, which is connected with the S-matrix by the relation: $S = (1 - iK)/(1 + iK)$ and apply the above cut-off operator in the calculation of the K-matrix [2].

Obviously the described method of relativistic cut-off can easily be formulated in all cases of collisions and constitutes a general theory of collisions and of non local interactions. In this theory the $\mathcal{G}^2$ operators have the significance of statistical weight-factors. These factors must be taken into account in order to evaluate the probabilities of different alternative (« channels ») we encounter in a collision process. One has to consider in general cases the interactions of the given set of incident particles with all kind of fields and one must introduce also non linear hamiltonians containing any number of creation operators. Since the number of coupling constants is presumably small, such more general hamiltonians must derive from a non linear type of interactions in accord with the ideas of Born and Heisenberg.

In the problem of the multiple production of mesons (as observed in « jets ») the present theory permits to apply a statistical treatment similar to the one suggested by Fermi and by the author, and to obtain meson spectra in accord with the theory of Heisenberg and the available experimental data.

The G operators defined above can be easily introduced also in the Tamm-Dancoff formalism, if one makes the calculations in the C-system (since in this case the G operators depend only on the space components of p_μ).

Markov and Yukawa suggested to consider non local fields depending on a new set of operators called « internal coordinates » η'_ν of a particle (besides the usual space-time and spin and isotopic spin coordinates (e.g. $\eta'_\nu = x_\nu - x^{(c)}_\nu$ where $x^{(c)}_\nu$ are the center of mass coordinates of the particle). In the case of a Dirac particle η'_ν could represent the unobservable « Zitterbewegung » and could depend on the operators γ^ν. It seems noteworthy that, from the covariance requirement, the representative of a pseudo-scalar meson field can depend only on the invariant $s = (\eta_\nu \eta^\nu)^{1/2}$. The canonically conjugated operator is the mass operator $(p_\nu p^\nu)^{1/2}$. In the case a Dirac particle the analogous operators are $\gamma^\nu \eta'_\nu$ and $\gamma^\nu p_\nu$.

The problem of the mass quantization seems to us independent from the question of non local interactions considered above. Let us consider the family of π-mesons. In order to give an example of a possible solution of the problem let us assume that the representatives are sum of products $\Phi(x_\nu)\varphi(s/l)$ and postulate that for the invariant $\varphi(s/l)$ an eigenvalue problem arises, described e.g. by the equation:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} + \varepsilon^2\right)\varphi(\varepsilon) = \mu\varphi(\varepsilon)\,,$$

where $\varepsilon = s/l$ and $\mu = ml$. This equation represents a kind of Zitterbewegung. One obtains a mass spectrum: $m = (2n + 1)1/l$. Similar solution (but different in many respects) bas been suggested by Yukawa [3].

A detailed account of the suggested method will be published elsewhere.

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[2] I am grateful to dr. S. Fubini for calling my attention on this point.

[3] Yukawa: *Phys. Rev.*, **91**, 415 (1953).

LIBRI RICEVUTI E RECENSIONI

OTTO GRAF – *Die wichtigsten Baustoffe des Hoch- u. Tiefbaus.* 16° Seiten 131 + VIII mit 63 Abbildungen Berlin 1953.

L'Autore, ingegnere e professore ordinario all'Università di Stoccarda, ha raccolto in un succinto manualetto le principali informazioni sui materiali di corrente impiego nelle costruzioni, desumendole dalle proprie pubblicazioni *Handbuch der Werkstoffprüfung* e *Die Baustoffe*.

In dodici brevi capitoletti vengono passati in rapida rassegna il legno, le pietre naturali, i leganti idraulici, i calcestruzzi e le malte sia di calce che cementizie, i materiali minori (gesso, magnesia, pozzolana, ecc.), i materiali per coperture, i materiali metallici ferrosi e quelli non ferrosi, il vetro, l'asfalto ed i prodotti sintetici per tinte, colle e simili. In ogni capitolo sono richiamate le norme DIN per l'accettazione dei materiali, e sono date indicazioni sui limiti d'impiego nonchè accenni alle caratteristiche di resistenza meccanica, agli agenti atmosferici fisici e chimici nelle diverse condizioni d'uso.

Per i principali materiali — legno, calcestruzzo, pietre, ferro — sono riportati grafici e tabelle relativamente alle più importanti qualità meccaniche.

Il manualetto è completato da un breve indice alfabetico delle voci, oltre all'indice sistematico.

G. SCACCIA SCARAFONI

ERRATA-CORRIGE

S. ALBERTONI e M. CUGIANI: **Sul problema del cambiamento di variabili nella teoria delle distribuzioni** (II), *Nuovo Cimento*, **10**, 157 (1953).

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171	15	dalla (12)	come in (11)

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